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RESEARCH REPORT
N° 8685
February 2015
Project-Team ACUMES

ISSN 0249-6399 ISRN INRIA/RR--8685--FR+ENG

The Lighthill-Whitham-Richards traffic flow model with non-local velocity: analytical study and numerical results

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Research Report n° 8685 — February 2015 — 36 pages

Abstract: We consider an extension of the classical Lighthill-Whitham-Richards traffic flow model, in which the mean velocity is assumed to be dependent on the downstream traffic density. We generalize the results of [4] to a general velocity function v, and propose a second order numerical scheme to compute approximate solutions.

Key-words: scalar conservation laws, non-local flux, macroscopic traffic flow models

This research was performed during the M2 internship of S. Scialanga at Inria from September to December 2014.

* Partially supported by the European Research Council under the European Union's Seventh Framework Program (FP/2007-2013) / ERC Grant Agreement n. 257661.
† Partially supported by the Faculty of Science of the University of Rome "Sapienza".

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Le modèle de trafic routier LWR avec vitesse non-locale: étude analytique et résultats numériques

Résumé : Nous considérons une extension du célèbre modèle Lighthill-Whitham-Richards de trafic routier, dans lequel la vitesse moyenne est supposée dépendre d'une moyenne pesée de la densité de trafic en aval. Nous généralisons les résultats de [4] à une fonction vitesse v générique et proposons un schéma numérique de deuxième ordre pour calculer les solutions approchées.

Mots-clés : lois de conservations scalaires, flux non-local, modèles de trafic macroscopiques

1 Introduction

We consider the mass conservation equation for traffic flow with non-local mean velocity: we assume that drivers adapt their velocity to the downstream traffic, assigning a greater importance to closer vehicles. We refer to the model proposed in [4], but for a general continuous decreasing velocity function. The density ρ satisfies the following mass conservation laws:

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) v\Big(\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\ dy \Big) \right) = 0\,, \tag{1}$$

defined for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, $\eta > 0$. The convolution kernel $w_\eta \in \mathbf{C}^1([0,\eta];\mathbb{R}^+)$ is a non-increasing function such that $\int_0^\eta w_\eta(x)dx = 1$ (for example, $w_\eta(x) \equiv 1/\eta$ or $w_\eta(x) = 2(1 - x/\eta)/\eta$).
In (1), we take a continuous non-increasing velocity function $v : [0, \rho_{\max}] \to \mathbb{R}^+$, such that:

$$-A \leq v' \leq 0, \quad \text{with } A \in \mathbb{R}^+,$$

$$v(0) = v_{\max}, \quad v(\rho_{\max}) = v_{\min} \geq 0\,.$$

We denote the downstream convolution product as

$$\rho *_d w_\eta(t,x) := \int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\ dy\,. \tag{2}$$

Setting $V(t,x) = v(\rho *_d w_\eta(t,x))$, we rewrite (1) as

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) V(t,x)\right) = 0\,,$$

with initial datum

$$\rho(0,x) = \rho_0(x) \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}]). \tag{3}$$

We will consider solutions $\rho = \rho(t,x)$ satisfying the following definitions (see [3, 4, 7, 10]):

Definition 1. A function $\rho \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathrm{BV})(\mathbb{R}^+ \times \mathbb{R};\mathbb{R})$ is a weak solution of (1), (3), if

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \left(\rho\varphi_t + \rho(t,x) v(\rho *_d w_\eta)\varphi_x\right) dxdt + \int_{-\infty}^{+\infty} \rho_0(x)\varphi(0,x)\ dx = 0 \tag{4}$$

$\forall \varphi \in \mathbf{C^1_c}(\mathbb{R}^2;\mathbb{R})$.

Definition 2. A function $\rho \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathrm{BV})(\mathbb{R}^+ \times \mathbb{R};\mathbb{R})$ is an entropy weak solution if

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \left(|\rho-\kappa|\varphi_t + |\rho-\kappa|\, V\, \varphi_x - \mathrm{sgn}(\rho-\kappa)\,\kappa\, V_x\ \varphi\right)(t,x) dxdt$$
$$+ \int_{-\infty}^{+\infty} |\rho_0(x) - \kappa| \varphi(0,x)\ dx \geq 0 \tag{5}$$

$\forall \varphi \in \mathbf{C^1_c}(\mathbb{R}^2;\mathbb{R}^+)$ and $\kappa \in \mathbb{R}$.

The main results are collected in the following theorem:

Theorem 1. *Let $\rho_0 \in BV(\mathbb{R};[0,\rho_{max}])$ and $w_\eta \in \mathbf{C^1}([0,\eta];\mathbb{R}^+)$ be a non-increasing function such that $\int_0^\eta w_\eta(x)dx = 1$. Then the Cauchy problem*

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho v(\rho *_d w_\eta)\right) = 0, & x \in \mathbb{R}, t > 0, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases}$$

admits a unique weak entropy solution in the sense of Definitions 1 and 2, such that

$$\min_{\mathbb{R}}\{\rho_0\} \leq \rho(t,x) \leq \max_{\mathbb{R}}\{\rho_0\}, \quad \textit{for a.e. } x \in \mathbb{R},\ t>0. \tag{6}$$

Uniqueness of solutions is proved in [4]. The proof of existence also follows closely [4], see also [1, 3, 6, 7, 8] for related results. In the following sections, we will describe the finite volume scheme used to construct approximate solutions and the proof of its fine proprieties: maximum principle, bounded total variation, discrete entropy inequalities and $\mathbf{L^1}$ stability estimate, as well as convergence to a weak entropy solution.

2 A Lax-Friedrichs numerical scheme

Let us consider a space step Δx such that $\eta = N\Delta x$, for some $N \in \mathbb{N}$, and a time step Δt subject to a CFL condition which will be specified later. For $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $x_{j+1/2} = j\Delta x$ be the cells interfaces, $x_j = (j-1/2)\Delta x$ the cells centers and $t^n = n\Delta t$ the time mesh. We aim at constructing a finite volume approximate solution $\rho_{\Delta x}(t,x) = \rho_j^n$ for $(t,x) \in C_j^n = [t^n, t^{n+1}[\times]x_{j-1/2}, x_{j+1/2}[$. Given the piece-wise constant approximation of the initial datum ρ_0,

$$\rho_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\ dx\,,$$

and denoting $w_\eta^k := w_\eta(k\Delta x)$ for $k = 0, \ldots, N-1$, we set

$$V_j := v\Big(\Delta x \sum_{k=0}^{N-1} w_\eta^k\ \rho_{j+k}\Big), \tag{7}$$

which involves a quadrature formula to approximate the convolution term. Remark that the above discretization chosen for w_η implies

$$\Delta x \sum_{k=0}^{N-1} w_\eta^k \leq 1 + w_\eta(0)\Delta x. \tag{8}$$

Remark. *Our choice of the discretization of the convolution integral is easy to implement but slightly underestimates traffic mean velocity and may introduce unphysical negative velocities. Other discretization choices are possible, see for example [3, eq. (3.2)].*

We consider the following modified Lax-Friedrichs flux adapted to (1):

$$\begin{aligned} F_{j+1/2}^n :=& g(\rho_j^n, \ldots, \rho_{j+N}^n) \\ =& \frac{1}{2}\rho_j^n V_j^n + \frac{1}{2}\rho_{j+1}^n V_{j+1}^n + \frac{\alpha}{2}(\rho_j^n - \rho_{j+1}^n)\ , \end{aligned} \tag{9}$$

where $\alpha \geq 1$ is the viscosity coefficient. This gives a $N+2$ points finite volume scheme

$$\rho_j^{n+1} = H(\rho_{j-1}^n, \ldots, \rho_{j+N}^n)\ , \tag{10}$$

where

$$\begin{aligned}
&H(\rho_{j-1},\dots,\rho_{j+N})\\
&:= \rho_j - \lambda\left(g(\rho_j,\dots,\rho_{j+N}) - g(\rho_{j-1},\dots,\rho_{j+N-1})\right)\\
&= \rho_j + \frac{\lambda\alpha}{2}\left(\rho_{j-1} - 2\rho_j + \rho_{j+1}\right) + \frac{\lambda}{2}\left(\rho_{j-1}V_{j-1} - \rho_{j+1}V_{j+1}\right),
\end{aligned}\tag{11}$$

with $\lambda = \Delta t/\Delta x$. In particular, we observe that $H(\rho,\dots,\rho) = \rho$ for all $\rho \in [0,\rho_{\max}]$.

Assuming $\rho_i \in [0,\rho_{\max}]$ for $i = j-1,\dots,j+N$, we compute the partial derivatives of H:

$$\begin{aligned}
\frac{\partial H}{\partial \rho_{j-1}} =& \frac{\lambda}{2}\Big(\alpha + V_{j-1} + \rho_{j-1}\frac{\partial V_{j-1}}{\partial \rho_{j-1}} - \rho_{j+1}\frac{\partial V_{j+1}}{\partial \rho_{j-1}}\Big)\\
=& \frac{\lambda}{2}\left(\alpha + v\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k \rho_{j-1+k}\Big) + \rho_{j-1}v'\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k \rho_{j-1+k}\Big)\Delta x w_\eta^0\right),
\end{aligned}\tag{12a}$$

$$\begin{aligned}
\frac{\partial H}{\partial \rho_{j}} =& 1 - \lambda\alpha + \frac{\lambda}{2}\Big(\rho_{j-1}\frac{\partial V_{j-1}}{\partial \rho_{j}} - \rho_{j+1}\frac{\partial V_{j+1}}{\partial \rho_{j}}\Big)\\
=& 1 - \lambda\alpha + \frac{\lambda}{2}\rho_{j-1}v'\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k\ \rho_{j-1+k}\Big)\Delta x w_\eta^1,
\end{aligned}\tag{12b}$$

$$\begin{aligned}
\frac{\partial H}{\partial \rho_{j+1}} =& \frac{\lambda}{2}\Big(\alpha + \rho_{j-1}\frac{\partial V_{j-1}}{\partial \rho_{j+1}} - V_{j+1} - \rho_{j+1}\frac{\partial V_{j+1}}{\partial \rho_{j+1}}\Big)\\
=& \frac{\lambda}{2}\Bigg(\alpha + \rho_{j-1}v'\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k \rho_{j-1+k}\Big)\Delta x w_\eta^2\\
&- \rho_{j+1}v'\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k\ \rho_{j+1+k}\Big)\Delta x w_\eta^0 - v\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k \rho_{j+1+k}\Big)\Bigg),
\end{aligned}\tag{12c}$$

$$\begin{aligned}
\frac{\partial H}{\partial \rho_{j+k}} =& \frac{\lambda}{2}\Delta x\Bigg(\rho_{j-1}w_\eta^{k+1}v'\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k \rho_{j-1+k}\Big)\\
&- w_\eta^{k-1}\rho_{j+1}v'\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k \rho_{j+1+k}\Big)\Bigg), \qquad k = 2,\dots,N-2
\end{aligned}\tag{12d}$$

$$\frac{\partial H}{\partial \rho_{j+N-1}} = -\frac{\lambda}{2}\Delta x w_\eta^{N-2}\rho_{j+1}v'\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k \rho_{j+1+k}\Big),\tag{12e}$$

$$\frac{\partial H}{\partial \rho_{j+N}} = -\frac{\lambda}{2}\Delta x w_\eta^{N-1}\rho_{j+1}v'\Big(\Delta x\sum_{k=0}^{N-1} w_\eta^k \rho_{j+1+k}\Big),\tag{12f}$$

Observe that (12e) and (12f) are non-negative since $v'(\cdot) \le 0$. Moreover, the CFL condition

$$\Delta t \le \frac{2}{2\alpha + A\Delta x\, w_\eta(0)}\Delta x\tag{13}$$

ensures the positivity of (12b) and the assumption

$$\alpha \ge v_{\max} + A\Delta x w_\eta(0)\tag{14}$$

guarantees the positivity of (12c) e (12a). To obtain (13) and (14) we used the fact that $w_\eta^k \le w_\eta(0)$ for $k = 0,\dots,N-1$, by non-increasing monotonicity of w_η. On the contrary, the sign of

(12d) cannot be a-priori determined. Therefore, the numerical scheme (9), (10) is not monotone, and classical convergence results do not apply.

2.1 Maximum principle and $\mathbf{L}^\infty$ estimates

We prove the following maximum principle property (see [4]).

Proposition 1. *For any initial datum ρ_j^0, $j \in \mathbb{Z}$, let*

$$\rho_m = \min_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}] \quad \textit{and} \quad \rho_M = \max_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]. \tag{15}$$

Then the finite volume approximation ρ_j^n, $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, constructed using scheme (9), (10) *satisfies the bounds*

$$\rho_m \le \rho_j^n \le \rho_M$$

for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, under the CFL condition (13).

The proof is a direct consequence of the following lemma.

Lemma 1. *Let $0 \le \rho_m \le \rho_j^n \le \rho_M \le \rho_{\max}$ $\forall\ j \in \mathbb{Z}$. Then*

$$H(\rho_m, \rho_m, \rho_m, \rho_{j+2}, \dots, \rho_{j+N-2}, \rho_m, \rho_m) \ge \rho_m, \tag{16}$$

$$H(\rho_M, \rho_M, \rho_M, \rho_{j+2}, \dots, \rho_{j+N-2}, \rho_M, \rho_M) \le \rho_M. \tag{17}$$

Proof. From (11) we get

$$\begin{aligned}
&H(\rho_m, \rho_m, \rho_m, \rho_{j+2}, \dots, \rho_{j+N-2}, \rho_m, \rho_m)\\
&= \rho_m + \frac{\lambda}{2}\rho_m\Big(V_{j-1} - V_{j+1}\Big)\\
&= \rho_m + \frac{\lambda}{2}\rho_m\left(v\Big(\Delta x \sum_{k=0}^{N-1} w_\eta^k\ \rho_{j-1+k}\Big) - v\Big(\Delta x \sum_{k=0}^{N-1} w_\eta^k\ \rho_{j+1+k}\Big)\right).
\end{aligned}$$

For the mean value theorem we get

$$v\Big(\Delta x \sum_{k=0}^{N-1} w_\eta^k\ \rho_{j-1+k}\Big) - v\Big(\Delta x \sum_{k=0}^{N-1} w_\eta^k\ \rho_{j+1+k}\Big) = v'(\xi)\left[\Delta x \sum_{k=0}^{N-1} w_\eta^k(\rho_{j-1+k} - \rho_{j+1+k})\right] \ge 0$$

for some ξ between $\Delta x \sum_{k=0}^{N-1} w_\eta^k\ \rho_{j-1+k}$ and $\Delta x \sum_{k=0}^{N-1} w_\eta^k\ \rho_{j+1+k}$. Indeed, we observe that

$$\begin{aligned}
&\sum_{k=0}^{N-1} w_\eta^k\,(\rho_{j-1+k} - \rho_{j+1-k})\\
&\qquad = \rho_m\left[w_\eta^0 + w_\eta^1 - w_\eta^{N-2} - w_\eta^{N-1}\right] + \sum_{k=1}^{N-2} \rho_{j+k}\left[w_\eta^{k+1} - w_\eta^{k-1}\right]\\
&\qquad \le \rho_m\left[w_\eta^0 + w_\eta^1 - w_\eta^{N-2} - w_\eta^{N-1}\right] + \rho_m \sum_{k=1}^{N-2}\left[w_\eta^{k+1} - w_\eta^{k-1}\right]\\
&\qquad = \rho_m\Big\{\sum_{k=-1}^{N-2} w_\eta^{k+1} - \sum_{k=1}^{N} w_\eta^{k-1}\Big\} = 0\ ,
\end{aligned}$$

where the inequality is due to the non-increasing monotonicity of w_η. Inequality (17) can be recovered following the same procedure. $\square$

Proof of Proposition 1
The proof is analogous as in [4], we recall it here for completeness. We apply the mean value theorem between the points $R_j^n = (\rho_{j-1}^n, \dots, \rho_{j+N}^n)$ and

$$R_m^n = (\rho_m, \rho_m, \rho_m, \rho_{j+2}^n, \dots, \rho_{j+N-2}^n, \rho_m, \rho_m),$$

which by (16) tell us

$$\begin{aligned} \rho_j^{n+1} = H(R_j^n) &= H(R_m^n) + \left\langle \nabla H(R_\xi), R_j^n - R_m^n \right\rangle \\ &\geq \rho_m + \left\langle \nabla H(R_\xi), R_j^n - R_m^n \right\rangle, \end{aligned} \tag{18}$$

for $R_\xi = (1-\xi)R_m^n + \xi R_j^n$, for some $\xi \in [0,1]$.
It is now enough to observe that

$$\frac{\partial H}{\partial \rho_{j+k}}(R_\xi)(R_j^n - R_m^n)_k = 0 \qquad k = 2, \dots, N-2,$$

since $(R_j^n - R_m^n)_k = 0$ for $k = 2, \dots, N-2$. Therefore, under the assumptions (13) and (14), we can conclude that $\left\langle \nabla H(R_\xi), R_j^n - R_m^n \right\rangle \geq 0$ and therefore by (18) we have proved that

$$\rho_j^{n+1} \geq \rho_m.$$

The upper bound $\rho_j^{n+1} \leq \rho_M$ is recovered similarly by considering

$$R_M^n = (\rho_M, \rho_M, \rho_M, \rho_{j+2}^n, \dots, \rho_{j+N-2}^n, \rho_M, \rho_M)$$

in place of R_m^n and using (17).

2.2 BV estimates

Accurate estimates show that the approximate solutions constructed using our numerical scheme have bounded total variation. This is the main result to prove the Theorem 1.

Proposition 2. *Let $\rho_0 \in BV(\mathbb{R}; [0, \rho_{\max}])$, and let $\rho_{\Delta x}$ be given by (10), (11). If $\alpha \geq v_{max} + A\, w_\eta(0)\Delta x$ (and $\alpha \geq 2A\, w_\eta(0)\Delta x$) and the CFL condition $\Delta t \leq \Delta x/(\alpha + 2A\, w_\eta(0)\Delta x)$) holds, then for every $T > 0$ the following discrete space BV holds*

$$TV(\rho_{\Delta x}(T, \cdot)) \leq C(w_\eta, v, \rho_0, T) := e^{w_\eta(0)\left(5A + 7\|v''\|_\infty\right)\frac{T}{2}}\, TV(\rho_0). \tag{19}$$

Proof. At the mesh cell C_j^n there holds

$$\rho_j^{n+1} = \rho_j + \frac{\lambda\alpha}{2}\left(\rho_{j-1} - 2\rho_j + \rho_{j+1}\right) + \frac{\lambda}{2}\left(\rho_{j-1}V_{j-1} - \rho_{j+1}V_{j+1}\right), \tag{20}$$

and at C_{j+1}^n

$$\rho_{j+1}^{n+1} = \rho_{j+1} + \frac{\lambda\alpha}{2}\left(\rho_j - 2\rho_{j+1} + \rho_{j+2}\right) + \frac{\lambda}{2}\left(\rho_j V_j - \rho_{j+2}V_{j+2}\right), \tag{21}$$

where for simplicity we have omitted the index n. Computing the difference between (21) and (20) and setting $\Delta_{j+k-1/2}^n = \rho_{j+k}^n - \rho_{j+k-1}^n$ for $k = 0, \dots, N+1$ we get:

$$\Delta_{j+\frac{1}{2}}^{n+1} = \Delta_{j+\frac{1}{2}} + \frac{\lambda\alpha}{2}\left[\Delta_{j-\frac{1}{2}} - 2\Delta_{j+\frac{1}{2}} + \Delta_{j+\frac{3}{2}}\right]$$

$$+\frac{\lambda}{2}\left[\rho_j V_j \pm \rho_{j-1} V_j - \rho_{j-1} V_{j-1} - \rho_{j+2} V_{j+2} \pm \rho_{j+1} V_{j+2} + \rho_{j+1} V_{j+1}\right] \tag{22}$$

adding and subtracting the red quantities.
We can rewrite (22) as:

$$\begin{aligned}\Delta_{j+\frac{1}{2}}^{n+1} &= \frac{\lambda\alpha}{2}\Delta_{j-\frac{1}{2}} + (1-\lambda\alpha)\,\Delta_{j+\frac{1}{2}} + \frac{\lambda\alpha}{2}\Delta_{j+\frac{3}{2}} \\ &+ \frac{\lambda}{2}\left[V_j\Delta_{j-\frac{1}{2}} + \rho_{j-1}\left(V_j - V_{j-1}\right) - V_{j+2}\Delta_{j+\frac{3}{2}} + \rho_{j+1}\left(V_{j+1} - V_{j+2}\right)\right].\end{aligned}$$

Remembering the definition of V_j and applying the main value theorem, we rewrite $V_j - V_{j-1}$ and $V_{j+2} - V_{j+1}$ as follow:

$$\begin{aligned}V_j - V_{j-1} &= v'(\xi)\Delta x\left[\sum_{k=0}^{N-1} w_\eta^k \rho_{j+k} - \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k-1}\right] = v'(\xi)\Delta x \sum_{k=0}^{N-1} w_\eta^k \Delta_{j+k-\frac{1}{2}} \\ V_{j+2} - V_{j+1} &= v'(\xi')\Delta x \sum_{k=0}^{N-1} w_\eta^k \Delta_{j+k+\frac{3}{2}}\end{aligned}$$

for some ξ between $\sum_{k=0}^{N-1} w_\eta^k \rho_{j+k}$ and $\sum_{k=0}^{N-1} w_\eta^k \rho_{j+k-1}$ and some ξ' between $\sum_{k=0}^{N-1} w_\eta^k \rho_{j+k+2}$ and $\sum_{k=0}^{N-1} w_\eta^k \rho_{j+k+1}$. Therefore:

$$\begin{aligned}\Delta_{j+\frac{1}{2}}^{n+1} &= \frac{\lambda}{2}\left(\alpha + V_j\right)\Delta_{j-\frac{1}{2}} + (1-\lambda\alpha)\Delta_{j+\frac{1}{2}} + \frac{\lambda}{2}\left(\alpha - V_{j+2}\right)\Delta_{j+\frac{3}{2}} \\ &+ \frac{\lambda}{2}\rho_{j-1}\Delta x\ v'(\xi)\sum_{k=0}^{N-1} w_\eta^k \Delta_{j+k-\frac{1}{2}} - \frac{\lambda}{2}\rho_{j+1}\Delta x\ v'(\xi')\sum_{k=0}^{N-1} w_\eta^k \Delta_{j+k+\frac{3}{2}}\end{aligned}$$

and we get:

$$\Delta_{j+\frac{1}{2}}^{n+1} = \frac{\lambda}{2}\left[\alpha + V_j + \rho_{j-1}\Delta x\ v'(\xi) w_\eta^0\right]\Delta_{j-\frac{1}{2}} \tag{23a}$$

$$+\left[1 - \lambda\alpha + \frac{\lambda}{2}\rho_{j-1}\Delta x\ v'(\xi) w_\eta^1\right]\Delta_{j+\frac{1}{2}} \tag{23b}$$

$$+\frac{\lambda}{2}\left[\alpha - V_{j+2} + \rho_{j-1}\Delta x\ v'(\xi) w_\eta^2 - \rho_{j+1}\Delta x\ v'(\xi') w_\eta^0\right]\Delta_{j+\frac{3}{2}} \tag{23c}$$

$$+\frac{\lambda}{2}\rho_{j-1}\Delta x\ v'(\xi)\sum_{k=3}^{N-1} w_\eta^k \Delta_{j+k-\frac{1}{2}} \tag{23d}$$

$$-\frac{\lambda}{2}\rho_{j+1}\Delta x\ v'(\xi')\sum_{k=1}^{N-1} w_\eta^k \Delta_{j+k+\frac{3}{2}}. \tag{23e}$$

Rearranging the indices in (23d) and (23e) we obtain:

$$\begin{aligned}(23\text{d}) + (23\text{e}) &= \frac{\lambda}{2}\ \Delta x \sum_{k=2}^{N-2}\left[\rho_{j-1} v'(\xi) w_\eta^{k+1} - \rho_{j+1} v'(\xi') w_\eta^{k-1}\right]\Delta_{j+k+\frac{1}{2}} \\ &- \frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w_\eta^{N-2}\Delta_{j+N-\frac{1}{2}} \\ &- \frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w_\eta^{N-1}\Delta_{j+N+\frac{1}{2}}.\end{aligned}$$

Noting that:

$$\begin{aligned}
&\rho_{j-1}v'(\xi)w_\eta^{k+1} - \rho_{j+1}v'(\xi')w_\eta^{k-1} \\
&\qquad = \rho_{j-1}\ v'(\xi)(w_\eta^{k+1} - w_\eta^{k-1}) && \text{(24a)}\\
&\qquad + w_\eta^{k-1}\left(\rho_{j-1}\ v'(\xi) \pm \rho_{j-1}\ v'(\xi') - \rho_{j+1}\ v'(\xi')\right) && \text{(24b)}
\end{aligned}$$

we rewrite

$$\begin{aligned}
\text{(24b)} &= w_\eta^{k-1}\left(\rho_{j-1}\left[v'(\xi) - v'(\xi')\right] + (\rho_{j-1} - \rho_{j+1})v'(\xi')\right)\\
&= w_\eta^{k-1}\left(\rho_{j-1}\left[v'(\xi) - v'(\xi')\right] + (\rho_{j-1} \pm \rho_j - \rho_{j+1})v'(\xi')\right)\\
&= w_\eta^{k-1}\left(\rho_{j-1}\left[v'(\xi) - v'(\xi')\right] - v'(\xi')\left[\Delta_{j-\frac{1}{2}} + \Delta_{j+\frac{1}{2}}\right]\right).
\end{aligned}$$

Therefore we get:

$$\begin{aligned}
\Delta_{j+\frac{1}{2}}^{n+1} = \frac{\lambda}{2}&\left[\alpha + V_j + \rho_{j-1}\Delta x\ v'(\xi)w_\eta^0 - \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}\Delta_{j+k+\frac{1}{2}}\right]\Delta_{j-\frac{1}{2}} && \text{(25a)}\\
+&\left[1 - \lambda\alpha + \frac{\lambda}{2}\rho_{j-1}\Delta x\ v'(\xi)w_\eta^1 - \frac{\lambda}{2}\Delta x\ v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}\Delta_{j+k+\frac{1}{2}}\right]\Delta_{j+\frac{1}{2}} && \text{(25b)}\\
+&\frac{\lambda}{2}\left[\alpha - V_{j+2} + \rho_{j-1}\Delta x\ v'(\xi)w_\eta^2 - \rho_{j+1}\Delta x\ v'(\xi')w_\eta^0\right]\Delta_{j+\frac{3}{2}} && \text{(25c)}\\
+&\frac{\lambda}{2}\Delta x\sum_{k=2}^{N-2}\Delta_{j+k+\frac{1}{2}}\left[\rho_{j-1}\ v'(\xi)(w_\eta^{k+1} - w_\eta^{k-1}) + w_\eta^{k-1}\rho_{j-1}\left(v'(\xi) - v'(\xi')\right)\right] && \text{(25d)}\\
-&\frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w_\eta^{N-2}\Delta_{j+N-\frac{1}{2}} && \text{(25e)}\\
-&\frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w_\eta^{N-1}\Delta_{j+N+\frac{1}{2}}. && \text{(25f)}
\end{aligned}$$

The coefficients of (25e) and (25f) are non-negative. The assumption $\alpha \geq 2A\Delta x\ w_\eta(0)$ guarantees the positivity of (25a), indeed if we require

$$\alpha + V_j + \rho_{j-1}\Delta x\ v'(\xi)w_\eta^0 - \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}\Delta_{j+k+\frac{1}{2}} \geq 0$$

$$\alpha \geq \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}\Delta_{j+k+\frac{1}{2}} - V_j - \rho_{j-1}\Delta x\ v'(\xi)w_\eta^0,$$

observing that

$$\begin{aligned}
-\rho_{j-1}\Delta x\ v'(\xi)w_\eta^0 &\leq\ \Delta x A\ w_\eta(0),\\
-V_j &\leq 0,
\end{aligned}$$

$$\begin{aligned}
\Delta x\ v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}\Delta_{j+k+\frac{1}{2}} &= \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}(\rho_{j+k+1} - \rho_{j+k})\\
&= -\Delta x\ v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}(\rho_{j+k} - \rho_{j+k+1})\\
&= -\Delta x\ v'(\xi')\left[\sum_{k=2}^{N-2} w_\eta^{k-1}\rho_{j+k} - \sum_{k=3}^{N-1} w_\eta^{k-2}\rho_{j+k}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\Delta x\ v'(\xi')\left[\sum_{k=3}^{N-2}(w_\eta^{k-1}-w_\eta^{k-2})\rho_{j+k}+w_\eta^1\rho_{j+2}-w_\eta^{N-3}\rho_{j+N-1}\right]\\
&\le -\Delta x\ v'(\xi')\left[\underbrace{\sum_{k=3}^{N-2}(w_\eta^{k-1}-w_\eta^{k-2})}_{=w_\eta^{N-3}-w_\eta^1}+w_\eta^1\right]\\
&= -\Delta x\ v'(\xi')w_\eta^{N-3}\\
&\le A\Delta x\ w_\eta(0),
\end{aligned}
$$

the condition on α follows. Similarly for (25c) we get $\alpha \geq v_{\max} + \Delta x A w_\eta(0)$ and for (25b) we get the following CFL condition

$$
\Delta t \le \frac{1}{\alpha + 2A\ \Delta x\ w_\eta(0)}\Delta x. \tag{26}
$$

Remark. *We underline that the coefficient in* (25d) *is not positive in general, hence the scheme is not monotonicity preserving unless* $v' = 0$ *as was the case in [4].*

Taking the absolute values in (25) and summing over $j \in \mathbb{Z}$ we get:

$$
\begin{aligned}
\sum_j \left|\Delta_{j+\frac12}^{n+1}\right| &\le \sum_j \frac{\lambda}{2}\left[\alpha + V_j + \rho_{j-1}\Delta x\ v'(\xi)w_\eta^0 - \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}\Delta_{j+k+\frac12}\right]\left|\Delta_{j-\frac12}\right|\\
&+\sum_j\left[1-\lambda\alpha+\frac{\lambda}{2}\rho_{j-1}\Delta x\ v'(\xi)w_\eta^1-\frac{\lambda}{2}\Delta x\ v'(\xi')\sum_{k=2}^{N-2}w_\eta^{k-1}\Delta_{j+k+\frac12}\right]\left|\Delta_{j+\frac12}\right|\\
&+\sum_j\frac{\lambda}{2}\left[\alpha - V_{j+2}+\rho_{j-1}\Delta x\ v'(\xi)w_\eta^2-\rho_{j+1}\Delta x\ v'(\xi')w_\eta^0\right]\left|\Delta_{j+\frac32}\right|\\
&+\sum_j\frac{\lambda}{2}\Delta x\ \sum_{k=2}^{N-2}\left|\Delta_{j+k+\frac12}\right|\left[\rho_{j-1}\ v'(\xi)(w_\eta^{k+1}-w_\eta^{k-1})+w_\eta^{k-1}\rho_{j-1}\left|v'(\xi)-v'(\xi')\right|\right]\\
&-\sum_j\frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w_\eta^{N-2}\left|\Delta_{j+N-\frac12}\right|\\
&-\sum_j\frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w_\eta^{N-1}\left|\Delta_{j+N+\frac12}\right|.
\end{aligned}\tag{27}
$$

Rearranging the indices we obtain:

$$
\begin{aligned}
&\sum_j\left|\Delta_{j+\frac12}^{n+1}\right|\\
&\le\sum_j\left|\Delta_{j+\frac12}\right|\Bigg[\frac{\lambda}{2}\left(\alpha+V_{j+1}+\rho_j\Delta x\ v'(\xi)w_\eta^0-\Delta x\ v'(\xi')\sum_{k=2}^{N-2}w_\eta^{k-1}\Delta_{j+k+\frac32}\right)\\
&+\left(1-\lambda\alpha+\frac{\lambda}{2}\rho_{j-1}\Delta x\ v'(\xi)w_\eta^1-\frac{\lambda}{2}\Delta x\ v'(\xi')\sum_{k=2}^{N-2}w_\eta^{k-1}\Delta_{j+k+\frac12}\right)\\
&+\frac{\lambda}{2}\left(\alpha-V_{j+1}+\rho_{j-2}\Delta x\ v'(\xi)w_\eta^2-\rho_j\Delta x\ v'(\xi')w_\eta^0\right)\\
&+\frac{\lambda}{2}\Delta x\sum_{k=2}^{N-2}\ \left(\rho_{j-k-1}\ v'(\xi)(w_\eta^{k+1}-w_\eta^{k-1})+w_\eta^{k-1}\rho_{j-k-1}\left|v'(\xi)-v'(\xi')\right|\right)
\end{aligned}
$$

$$
- \frac{\lambda}{2}\rho_{j-N+2}\ \Delta x\ v'(\xi')\ w_\eta^{N-2} - \frac{\lambda}{2}\rho_{j-N+1}\ \Delta x\ v'(\xi')\ w_\eta^{N-1}\Bigg]. \tag{28}
$$

Noting that

$$
\begin{aligned}
&- v'(\xi') \sum_{k=2}^{N-2} w_\eta^{k-1}\Delta_{j+k+\frac{3}{2}} - v'(\xi') \sum_{k=2}^{N-2} w_\eta^{k-1}\Delta_{j+k+\frac{1}{2}} \\
&= -v'(\xi')\left[\sum_{k=3}^{N-1} \Delta_{j+k+\frac{1}{2}} w_\eta^{k-2} + \sum_{k=2}^{N-2} \Delta_{j+k+\frac{1}{2}} w_\eta^{k-1}\right] \\
&= -v'(\xi')\left[\sum_{k=3}^{N-2} (w_\eta^{k-2} + w_\eta^{k-1})\Delta_{j+k+\frac{1}{2}} + \Delta_{j+N-\frac{1}{2}} w_\eta^{N-3} + \Delta_{j+\frac{5}{2}} w_\eta^{1}\right] \\
&= -v'(\xi')\Bigg[\sum_{k=4}^{N-1} \rho_{j+k}(w_\eta^{k-3} + w_\eta^{k-2}) - \sum_{k=3}^{N-2} \rho_{j+k}(w_\eta^{k-2} + w_\eta^{k-1}) \\
&\quad + w_\eta^{N-3}(\rho_{j+N} - \rho_{j+N-1}) + w_\eta^{1}(\rho_{j+3} - \rho_{j+2})\Bigg] \\
&= -v'(\xi')\Bigg[\sum_{k=4}^{N-2} (w_\eta^{k-3} + w_\eta^{k-2} - w_\eta^{k-2} - w_\eta^{k-1})\rho_{j+k} + \rho_{j+N-1}(w_\eta^{N-4} + w_\eta^{N-3}) \\
&\quad - \rho_{j+3}(w_\eta^{1} + w_\eta^{2}) + w_\eta^{N-3}(\rho_{j+N} - \rho_{j+N-1}) + w_\eta^{1}(\rho_{j+3} - \rho_{j+2})\Bigg],
\end{aligned}
$$

due to some cancellations, the coefficient of the right-hand side of (28) becomes

$$
\begin{aligned}
1 + \frac{\Delta t}{2}\Bigg\{&\rho_j w_\eta^0\Big(v'(\xi) - v'(\xi')\Big) + \rho_{j-1} v'(\xi) w_\eta^1 + \rho_{j-2} v'(\xi) w_\eta^2 \\
&-v'(\xi')\Bigg[\sum_{k=4}^{N-2}\Big(w_\eta^{k-3} - w_\eta^{k-1}\Big)\rho_{j+k} + \rho_{j+N-1} w_\eta^{N-4} \\
&\qquad + \rho_{j+N} w_\eta^{N-3} - \rho_{j+3} w_\eta^2 - \rho_{j-2} w_\eta^1\Bigg] \\
&+ \sum_{k=2}^{N-2}\Bigg[v'(\xi)\rho_{j-k-1}\Big(w_\eta^{k+1} - w_\eta^{k-1}\Big) + w_\eta^{k-1}\rho_{j-k-1}|v'(\xi) - v'(\xi')|\Bigg] \\
&-v'(\xi')\rho_{j-N+2} w_\eta^{N-2} - v'(\xi')\rho_{j-N+1} w_\eta^{N-1}\Bigg\}.
\end{aligned} \tag{29}
$$

We rewrite $v'(\xi) - v'(\xi')$ as $v''(\beta)(\xi - \xi')$, with β a point between ξ and ξ', and ξ, ξ' linear combinations, with $0 \le \theta, \mu \le 1$, s.t.:

$$
\begin{aligned}
\xi - \xi' &= \Delta x\Bigg\{\theta \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k} + (1-\theta) \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k-1} - \mu \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k+2} - (1-\mu) \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k+1}\Bigg\} \\
&= \Delta x\Bigg\{\theta \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k} + (1-\theta) \sum_{k=-1}^{N-2} w_\eta^{k+1} \rho_{j+k} - \mu \sum_{k=2}^{N+1} w_\eta^{k-2} \rho_{j+k} - (1-\mu) \sum_{k=1}^{N} w_\eta^{k-1} \rho_{j+k}\Bigg\} \\
&= \Delta x\Bigg\{\sum_{k=2}^{N-2}\Big[\theta w_\eta^k + (1-\theta) w_\eta^{k+1} - \mu w_\eta^{k-2} - (1-\mu) w_\eta^{k-1}\Big]\rho_{j+k} \\
&\quad + \theta\left(w_\eta^0 \rho_j + w_\eta^1 \rho_{j+1} + w_\eta^{N-1}\rho_{j+N-1}\right) + (1-\theta)\left(w_\eta^0 \rho_{j-1} + w_\eta^1 \rho_j + w_\eta^2 \rho_{j+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \mu \left(w_\eta^{N-3}\rho_{j+N-1} + w_\eta^{N-2}\rho_{j+N} + w_\eta^{N-1}\rho_{j+N+1}\right) \\
&- (1-\mu)\left(w_\eta^{0}\rho_{j+1} + w_\eta^{N-2}\rho_{j+N-1} + w_\eta^{N-1}\rho_{j+N}\right)\Big\}.
\end{aligned}
$$

Since by monotonicity of w_η

$$
\theta w_\eta^k + (1-\theta)w_\eta^{k+1} - \mu w_\eta^{k-2} - (1-\mu)w_\eta^{k-1} \le 0,
$$

taking the absolute values we get

$$
\begin{aligned}
|\xi - \xi'| \le\ & \Delta x \sum_{k=2}^{N-2}\Big(\mu w_\eta^{k-2} + (1-\mu)w_\eta^{k-1} - \theta w_\eta^k - (1-\theta)w_\eta^{k+1}\Big)\rho_{j+k} + \underbrace{w_\eta^0 + w_\eta^1 + w_\eta^2 + w_\eta^{N-1}}_{\le 4w_\eta^0} \\
\le\ & \Delta x\Big\{\sum_{k=2}^{N-2}(w_\eta^{k-2} - w_\eta^{k+1}) + 4w_\eta^0\Big\} \\
=\ & \Delta x\Big\{\sum_{k=0}^{N-4} w_\eta^k - \sum_{k=3}^{N-1} w_\eta^k + 4w_\eta^0\Big\} \\
=\ & \Delta x\{w_\eta^0 + w_\eta^1 + w_\eta^2 - w_\eta^{N-3} - w_\eta^{N-2} - w_\eta^{N-1} + 4w_\eta^0\} \\
\le\ & 7w_\eta^0\ \Delta x\,,
\end{aligned}
$$

where we have used the monotonicity of $w_\eta(x)$ ($w_\eta^k \le w_\eta(0)$). Therefore

$$
|v'(\xi) - v'(\xi')| \le 7\|v''\|_\infty w_\eta(0)\ \Delta x \tag{30}
$$

and

$$
\begin{aligned}
(29) \le\ & 1 + \frac{\Delta t}{2}A\Big\{w_\eta^0 + \underbrace{\sum_{k=4}^{N-2}(w_\eta^{k-3} - w_\eta^{k-1})}_{=\sum_1^{N-5} w_\eta^k - \sum_{k=3}^{N-3} w_\eta^k} + w_\eta^{N-4} + w_\eta^{N-3} \\
& + \underbrace{\sum_{k=2}^{N-2}(w_\eta^{k-1} - w_\eta^{k+1})}_{=\sum_{k=1}^{N-3} w_\eta^k - \sum_{k=3}^{N-1} w_\eta^k} + \frac{7w_\eta^0}{A}\|v''\|_\infty\ \underbrace{\Delta x\sum_{k=2}^{N-2} w_\eta^{k-1}}_{=\Delta x\sum_{k=1}^{N-3} w_\eta^k \le 1} + w_\eta^{N-2} + w_\eta^{N-1}\Big\} \\
\le\ & 1 + \frac{\Delta t}{2}A\Big\{w_\eta^0 + w_\eta^1 + w_\eta^2 - w_\eta^{N-4} - w_\eta^{N-3} + w_\eta^{N-4} + w_\eta^{N-3} \\
& + w_\eta^1 + w_\eta^2 - w_\eta^{N-2} - w_\eta^{N-1} + w_\eta^{N-2} + w_\eta^{N-1} + \frac{7w_\eta^0}{A}\|v''\|_\infty\Big\} \\
\le\ & 1 + \frac{\Delta t}{2}A\Big(5w_\eta^0 + \frac{7w_\eta^0}{A}\|v''\|_\infty\Big).
\end{aligned}
$$

Substituting in (28) we get

$$
\sum_j \left|\Delta_{j+\frac12}^{n+1}\right| \le \left[1 + \frac{\Delta t}{2}A\Big(5w_\eta^0 + \frac{7w_\eta^0}{A}\|v''\|_\infty\Big)\right]\sum_j \left|\Delta_{j+\frac12}^{n}\right|.
$$

Therefore we recover the following estimate for the total variation

$$\begin{aligned}\mathrm{TV}(\rho_{\Delta x}(T,\cdot)) &\le \left[1+\frac{\Delta t}{2}w_\eta(0)\Big(5A+7\|v''\|_\infty\Big)\right]^{T/\Delta t}\mathrm{TV}(\rho_{\Delta x}(0,\cdot))\\ &\le e^{w_\eta(0)\left(5A+7\|v''\|_\infty\right)\frac{T}{2}}\mathrm{TV}(\rho_0)\ .\end{aligned}$$

□

Moreover we recall the following [4]:

Corollary 1. *Let $\rho_0 \in BV(\mathbb{R};[0,1])$, and let $\rho_{\Delta x}$ given by* (10), (11)*. If $\alpha \ge 2A\ w_\eta(0)\Delta x$, $\alpha \ge v_{max} + A\ w_\eta(0)\Delta x$ and $\Delta t \le \Delta x/(\alpha + 2A\ w_\eta(0)\Delta x)$, then for every $T > 0$ exists $\widetilde{C} = \widetilde{C}(w_\eta, \rho_0, T, \alpha)$ such that*

$$TV(\rho_{\Delta x};[0,T]\times\mathbb{R}) \le \widetilde{C}. \tag{31}$$

2.3 Discrete entropy inequalities

We recall [4, Section 3.3] for completeness. Following [1, Proposition 2.8], we derive a discrete entropy inequality for the approximate solution generate by (10), (11). Let us denote by

$$\begin{aligned}G_{j+1/2}(u,v) &= \frac{1}{2}u\,V_j^n + \frac{1}{2}v\,V_{j+1}^n + \frac{\alpha}{2}(u-v)\ ,\\ F^\kappa_{j+1/2}(u,v) &= G_{j+1/2}(u\wedge\kappa, v\wedge\kappa) - G_{j+1/2}(u\vee\kappa, v\vee\kappa),\end{aligned}$$

with $a\wedge b = \max(a,b)$ and $a \vee b = \min(a,b)$.

Proposition 3. *Let ρ_j^n, $j\in\mathbb{Z}$, $n\in\mathbb{N}$, be given by* (9), (10)*. Then, if $\alpha \ge 1$ and the CFL condition* (13) *holds, for all $j\in\mathbb{Z}$, $n\in\mathbb{N}$ we have*

$$\begin{aligned}\left|\rho_j^{n+1}-\kappa\right| - \left|\rho_j^n-\kappa\right| + \lambda\left(F^\kappa_{j+1/2}(\rho_j^n,\rho_{j+1}^n) - F^\kappa_{j-1/2}(\rho_{j-1}^n,\rho_j^n)\right)&\\ +\frac{\lambda}{2}\,\mathrm{sgn}(\rho_j^{n+1}-\kappa)\ \kappa\left(V_{j+1}^n - V_{j-1}^n\right) \le 0&\end{aligned} \tag{32}$$

for all $\kappa\in\mathbb{R}$.

Proof. Setting

$$\tilde{H}_j(u,v,z) = v - \lambda\left(G_{j+1/2}(v,z) - G_{j-1/2}(u,v)\right)\ ,$$

the function $\tilde{H}_j$ is monotone non-decreasing in its first variable, monotone non-decreasing in its second variable for $\alpha\lambda \le 1$, which is guaranteed by the CFL condition (13), and monotone non-decreasing in its third variable for $\alpha \ge 1$, which is guaranteed by (14). Moreover, we have the identity

$$\begin{aligned}&\tilde{H}_j(\rho_{j-1}^n\wedge\kappa, \rho_j^n\wedge\kappa, \rho_{j+1}^n\wedge\kappa) - \tilde{H}_j(\rho_{j-1}^n\vee\kappa, \rho_j^n\vee\kappa, \rho_{j+1}^n\vee\kappa)\\ &\qquad= \left|\rho_j^n-\kappa\right| - \lambda\left(F^\kappa_{j+1/2}(\rho_j^n,\rho_{j+1}^n) - F^\kappa_{j-1/2}(\rho_{j-1}^n,\rho_j^n)\right).\end{aligned}$$

By monotonicity,

$$\tilde{H}_j(\rho_{j-1}^n\wedge\kappa, \rho_j^n\wedge\kappa, \rho_{j+1}^n\wedge\kappa) - \tilde{H}_j(\rho_{j-1}^n\vee\kappa, \rho_j^n\vee\kappa, \rho_{j+1}^n\vee\kappa)$$

$$
\begin{aligned}
&=\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)\wedge\tilde{H}_j(\kappa,\kappa,\kappa)-\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)\vee\tilde{H}_j(\kappa,\kappa,\kappa)\\
&=\left|\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\tilde{H}_j(\kappa,\kappa,\kappa)\right|\\
&=\operatorname{sgn}\left(\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\tilde{H}_j(\kappa,\kappa,\kappa)\right)\left(\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\tilde{H}_j(\kappa,\kappa,\kappa)\right)\\
&=\operatorname{sgn}\left(\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\kappa+\frac{\lambda}{2}\kappa\left(V_{j+1}^n-V_{j-1}^n\right)\right)\\
&\quad\left(\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\kappa+\frac{\lambda}{2}\kappa\left(V_{j+1}^n-V_{j-1}^n\right)\right)\\
&\geq\operatorname{sgn}\left(\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\kappa\right)\left(\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\kappa+\frac{\lambda}{2}\kappa\left(V_{j+1}^n-V_{j-1}^n\right)\right)\\
&=\left|\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\kappa\right|+\frac{\lambda}{2}\operatorname{sgn}\left(\tilde{H}_j(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)-\kappa\right)\kappa\left(V_{j+1}^n-V_{j-1}^n\right)\\
&=\left|\rho_j^{n+1}-\kappa\right|+\frac{\lambda}{2}\operatorname{sgn}\left(\rho_j^{n+1}-\kappa\right)\kappa\left(V_{j+1}^n-V_{j-1}^n\right),
\end{aligned}
$$

by definition of the scheme (10), (11), which gives (32). □

2.4 $\mathbf{L^1}$ stability estimates

We prove explicit $\mathbf{L^1}$ estimates that ensure the stability of the scheme (10), (11).

Proposition 4. *Let $\rho_0,\bar{\rho}_0\in BV(\mathbb{R};[0,\rho_{\max}])$ be two initial data, and denote by $\rho_{\Delta x},\bar{\rho}_{\Delta x}$ the corresponding approximate solutions constructed applying the modified Lax-Friedrichs scheme* (10), (11)*:*

$$
\rho_j^{n+1}=\rho_j^n+\frac{\lambda\alpha}{2}\left(\rho_{j-1}^n-2\rho_j^n+\rho_{j+1}^n\right)+\frac{\lambda}{2}\left(\rho_{j-1}^nV_{j-1}^n-\rho_{j+1}V_{j+1}^n\right)\ , \tag{33}
$$

$$
\bar{\rho}_j^{n+1}=\bar{\rho}_j^n+\frac{\lambda\alpha}{2}\left(\bar{\rho}_{j-1}^n-2\bar{\rho}_j^n+\bar{\rho}_{j+1}^n\right)+\frac{\lambda}{2}\left(\bar{\rho}_{j-1}^n\bar{V}_{j-1}^n-\bar{\rho}_{j+1}\bar{V}_{j+1}^n\right)\ , \tag{34}
$$

where we have set $V_j^n=v(\Delta x\sum_{k=0}^{N-1}w_\eta^k\ \rho_{j+k}^n)$ and $\bar{V}_j^n=v(\Delta x\sum_{k=0}^{N-1}w_\eta^k\ \bar{\rho}_{j+k}^n)$. Then, under assumptions (13) *and* (14)*, the following estimate holds:*

$$
\|\rho_{\Delta x}(T,\cdot)-\bar{\rho}_{\Delta x}(T,\cdot)\|_{\mathbf{L^1}}\leq K(w_\eta,\rho_0,\bar{\rho}_0,T)\|\rho_0-\bar{\rho}_0\|_{\mathbf{L^1}} \tag{35}
$$

with

$$
K(w_\eta,\rho_0,\bar{\rho}_0,T)=\exp\left(w_\eta(0)\big(3A+7\|v''\|_\infty+A\ \ TV(\rho_{\Delta x}(T))\big)\frac{T}{2}\right).
$$

Proof. Subtracting (34) from (33) we get

$$
\begin{aligned}
&\rho_j^{n+1}-\bar{\rho}_j^{n+1}=\\
&\quad=(1-\lambda\alpha)(\rho_j^n-\bar{\rho}_j^n)+\frac{\lambda\alpha}{2}(\rho_{j-1}^n-\bar{\rho}_{j-1}^n)+\frac{\lambda\alpha}{2}(\rho_{j+1}^n-\bar{\rho}_{j+1}^n)\\
&\qquad+\frac{\lambda}{2}\left(\rho_{j-1}^n(V_{j-1}^n-\bar{V}_{j-1}^n)+(\rho_{j-1}^n-\bar{\rho}_{j-1}^n)\bar{V}_{j-1}^n\right)\\
&\qquad-\frac{\lambda}{2}\left((\rho_{j+1}^n-\bar{\rho}_{j+1}^n)\bar{V}_{j+1}^n+\rho_{j+1}^n(V_{j+1}^n-\bar{V}_{j+1}^n)\right)\\
&\quad=(1-\lambda\alpha)(\rho_j^n-\bar{\rho}_j^n)+\frac{\lambda\alpha}{2}(\rho_{j-1}^n-\bar{\rho}_{j-1}^n)+\frac{\lambda\alpha}{2}(\rho_{j+1}^n-\bar{\rho}_{j+1}^n)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda}{2}\left[\rho_{j-1}^n v'(\xi)\Delta x\sum_{k=0}^{N-1} w_\eta^k(\rho_{j+k-1}^n-\bar\rho_{j+k-1}^n)+(\rho_{j-1}^n-\bar\rho_{j-1}^n)\bar V_{j-1}^n\right]\\
&-\frac{\lambda}{2}\left[(\rho_{j+1}^n-\bar\rho_{j+1}^n)\bar V_{j+1}^n+\rho_{j+1}v'(\xi')\Delta x\sum_{k=0}^{N-1} w_\eta^k(\rho_{j+k+1}^n-\bar\rho_{j+k+1}^n)\right]\\
=&\ \frac{\lambda}{2}\left[\alpha+\bar V_{j-1}+\rho_{j-1}v'(\xi)\Delta x\ w_\eta^0\right](\rho_{j-1}-\bar\rho_{j-1})\\
&+\left[1-\lambda\alpha+\frac{\lambda}{2}\rho_{j-1}v'(\xi)\Delta x\ w_\eta^1\right](\rho_j-\bar\rho_j)\\
&+\frac{\lambda}{2}\left[\alpha-\bar V_{j+1}+\rho_{j-1}v'(\xi)\Delta x\ w_\eta^2-\rho_{j+1}v'(\xi')\Delta x\ w_\eta^0\right](\rho_{j+1}-\bar\rho_{j+1})\\
&+\frac{\lambda}{2}\left[\Delta x\left(\rho_{j-1}v'(\xi)\sum_{k=2}^{N-2} w_\eta^{k+1}-\rho_{j+1}v'(\xi')\sum_{k=2}^{N} w_\eta^{k-1}\right)\right](\rho_{j+k}-\bar\rho_{j+k})\\
=&\ \frac{\lambda}{2}\left[\alpha+\bar V_{j-1}+\rho_{j-1}v'(\xi)\Delta x\ w_\eta^0\right](\rho_{j-1}-\bar\rho_{j-1})\\
&+\left[1-\lambda\alpha+\frac{\lambda}{2}\rho_{j-1}v'(\xi)\Delta x\ w_\eta^1\right](\rho_j-\bar\rho_j)\\
&+\frac{\lambda}{2}\left[\alpha-\bar V_{j+1}+\rho_{j-1}v'(\xi)\Delta x\ w_\eta^2-\rho_{j+1}v'(\xi')\Delta x\ w_\eta^0\right](\rho_{j+1}-\bar\rho_{j+1})\\
&+\frac{\lambda}{2}\Delta x\sum_{k=2}^{N-2}\left(\rho_{j-1}v'(\xi)w_\eta^{k+1}-\rho_{j+1}v'(\xi')w_\eta^{k-1}\right)(\rho_{j+k}-\bar\rho_{j+k})\\
&-\frac{\lambda}{2}\Delta x\ \rho_{j+1}v'(\xi')w_\eta^{N-2}(\rho_{j+N-1}-\bar\rho_{j+N-1})\\
&-\frac{\lambda}{2}\Delta x\ \rho_{j+1}v'(\xi')w_\eta^{N-1}(\rho_{j+N}-\bar\rho_{j+N}).
\end{aligned}
$$

Observe that the coefficient of the second term is positive thanks to (13) and the coefficients of the first and third terms are positive thanks to (14). Therefore, taking the absolute values in the above equality we get

$$
\begin{aligned}
&|\rho_j^{n+1}-\bar\rho_j^{n+1}|\\
&\le\frac{\lambda}{2}\left[\alpha+\bar V_{j-1}+\rho_{j-1}v'(\xi)\Delta x\ w_\eta^0\right]|\rho_{j-1}-\bar\rho_{j-1}|\\
&\quad+\left[1-\lambda\alpha+\frac{\lambda}{2}\rho_{j-1}v'(\xi)\Delta x\ w_\eta^1\right]|\rho_j-\bar\rho_j|\\
&\quad+\frac{\lambda}{2}\left[\alpha-\bar V_{j+1}+\rho_{j-1}v'(\xi)\Delta x\ w_\eta^2-\rho_{j+1}v'(\xi')\Delta x\ w_\eta^0\right]|\rho_{j+1}-\bar\rho_{j+1}|\\
&\quad+\frac{\lambda}{2}\Delta x\sum_{k=2}^{N-2}\left|\rho_{j-1}v'(\xi)w_\eta^{k+1}-\rho_{j+1}v'(\xi')w_\eta^{k-1}\right||\rho_{j+k}-\bar\rho_{j+k}|\\
&\quad-\frac{\lambda}{2}\Delta x\ \rho_{j+1}v'(\xi')w_\eta^{N-2}|\rho_{j+N-1}-\bar\rho_{j+N-1}|\\
&\quad-\frac{\lambda}{2}\Delta x\ \rho_{j+1}v'(\xi')w_\eta^{N-1}|\rho_{j+N}-\bar\rho_{j+N}|
\end{aligned}
$$

Summing over $j\in\mathbb{Z}$, rearranging the indexes and observing that

$$
\left|\rho_{j-1}v'(\xi)w_\eta^{k+1}-\rho_{j+1}v'(\xi')w_\eta^{k-1}\right|
$$

$$
\begin{aligned}
&\leq \left|\rho_{j-1}v'(\xi)(w_\eta^{k-1}-w_\eta^{k+1})\right| + w_\eta^{k-1}|\rho_{j-1}v'(\xi)-\rho_{j+1}v'(\xi')| \\
&= -\rho_{j-1}v'(\xi)(w_\eta^{k-1}-w_\eta^{k+1}) + w_\eta^{k-1}\Big|\rho_{j-1}\left[v'(\xi)-v'(\xi')\right] + (\rho_{j-1}-\rho_{j+1})v'(\xi')\Big|,
\end{aligned}
$$

thanks to some cancellations, we get

$$
\begin{aligned}
&\sum_j |\rho_j^{n+1}-\bar\rho_j^{n+1}| \\
&\quad\leq \sum_j \frac{\lambda}{2}\left[\alpha+\bar V_{j-1}+\rho_{j-1}v'(\xi)\Delta x\ w_\eta^0\right]|\rho_{j-1}-\bar\rho_{j-1}| \\
&\qquad+ \sum_j\left[1-\lambda\alpha+\frac{\lambda}{2}\rho_{j-1}v'(\xi)\Delta x\ w_\eta^1\right]|\rho_j-\bar\rho_j| \\
&\qquad+ \sum_j \frac{\lambda}{2}\left[\alpha-\bar V_{j+1}+\rho_{j-1}v'(\xi)\Delta x\ w_\eta^2-\rho_{j+1}v'(\xi')\Delta x\ w_\eta^0\right]|\rho_{j+1}-\bar\rho_{j+1}| \\
&\qquad- \sum_j\frac{\lambda}{2}\Delta x\ v'(\xi)\rho_{j-1}\sum_{k=2}^{N-2}(w_\eta^{k-1}-w_\eta^{k+1})|\rho_{j+k}-\bar\rho_{j+k}| \\
&\qquad+ \sum_j\frac{\lambda}{2}\Delta x\ \rho_{j-1}\sum_{k=2}^{N-2}w_\eta^{k-1}|v'(\xi)-v'(\xi')||\rho_{j+k}-\bar\rho_{j+k}| \\
&\qquad- \sum_j\frac{\lambda}{2}\Delta x\ v'(\xi')\sum_{k=2}^{N-2}w_\eta^{k-1}|\rho_{j-1}-\rho_{j+1}||\rho_{j+k}-\bar\rho_{j+k}| \\
&\qquad- \sum_j\frac{\lambda}{2}\Delta x\ \rho_{j+1}v'(\xi')w_\eta^{N-2}|\rho_{j+N-1}-\bar\rho_{j+N-1}| \\
&\qquad- \sum_j\frac{\lambda}{2}\Delta x\ \rho_{j+1}v'(\xi')w_\eta^{N-1}|\rho_{j+N}-\bar\rho_{j+N}| \\
&\quad\leq \sum_j|\rho_j^n-\bar\rho_j^n|\Bigg[1+\frac{\lambda}{2}\Bigg(\rho_j v'(\xi)\Delta x\ w_\eta^0+\rho_{j-1}\ v'(\xi)\Delta x\ w_\eta^1 \\
&\qquad+\rho_{j-2}\ v'(\xi)\Delta x\ w_\eta^2-\rho_j v'(\xi')\Delta x\ w_\eta^0-\Delta x\ v'(\xi)\sum_{k=2}^{N-2}\rho_{j-k-1}(w_\eta^{k-1}-w_\eta^{k+1}) \\
&\qquad+7\Delta x\|v''\|_\infty w_\eta(0)\Delta x\sum_{k=2}^{N-2}w_\eta^{k-1}\rho_{j-k-1}-\Delta x\sum_{k=2}^{N-2}w_\eta^{k-1}v'(\xi')|\rho_{j-k-1}-\rho_{j-k+1}| \\
&\qquad-\frac{\lambda}{2}\Delta x\ \rho_{j-N+2}v'(\xi')w_\eta^{N-2}-\frac{\lambda}{2}\Delta x\ \rho_{j-N+1}v'(\xi')w_\eta^{N-1}\Bigg)\Bigg].
\end{aligned}
$$

The coefficient of the above expression is bounded by

$$
\begin{aligned}
&\bigg[1+\frac{\Delta t}{2}\Big(Aw_\eta^0+Aw_\eta^{N-2}+Aw_\eta^{N-1}+7\big\|v''\big\|_\infty w_\eta(0) \\
&\qquad+A\big(w_\eta^1+w_\eta^2-w_\eta^{N-2}-w_\eta^{N-1}\big)+Aw_\eta(0)\mathrm{TV}(\rho_{\Delta x}(t^n,\cdot))\Big)\bigg] \\
&\leq\bigg[1+\frac{\Delta t}{2}\Big(3Aw_\eta(0)+7\big\|v''\big\|_\infty w_\eta(0)+Aw_\eta(0)\mathrm{TV}(\rho_{\Delta x}(t^n,\cdot))\Big)\bigg].
\end{aligned}
$$

Therefore

$$\begin{aligned}&\left\|\rho_{\Delta x}(T,\cdot)-\bar{\rho}_{\Delta x}(T,\cdot)\right\|_{\mathbf{L^1}}\leq\\&\leq\left[1+\frac{\Delta t}{2}\Big(3Aw_\eta(0)+7\left\|v''\right\|_\infty w_\eta(0)+Aw_\eta(0)\mathrm{TV}(\rho_{\Delta x}(T))\Big)\right]^{T/\Delta t}\left\|\rho_0-\bar{\rho}_0\right\|_{\mathbf{L^1}},\end{aligned}$$

which gives the desired estimate (35). □

2.5 Proof of Theorem 1

The proof follows closely [4, Section 4]. We reproduce it here in details for completeness. By Proposition 1 and Corollary 1, we can apply Helly's theorem, stating that there exists a subsequence, still denoted by $\rho_{\Delta x}$, that converges to some $\rho \in (\mathbf{L^1}\cap\mathbf{L^\infty}\cap\mathrm{BV})(\mathbb{R}^+\times\mathbb{R};[0,\rho_{\max}])$ in the $\mathbf{L^1_{loc}}$-norm.

Let $\varphi \in \mathbf{C^1_c}(\mathbb{R}^2)$ and multiply (10) by $\varphi(t^n,x_j)$. Summing over $j\in\mathbb{Z}$ and $n\in\mathbb{N}$ we get

$$\begin{aligned}&\sum_n\sum_j\varphi(t^n,x_j)\left(\rho_j^{n+1}-\rho_j^n\right)\\&=-\lambda\sum_n\sum_j\varphi(t^n,x_j)\left(g(\rho_j^n,\ldots,\rho_{j+N}^n)-g(\rho_{j-1}^n,\ldots,\rho_{j+N-1}^n)\right).\end{aligned}$$

Summing by parts we obtain

$$\begin{aligned}\sum_j\varphi(0,x_j)\rho_j^0&+\sum_n\sum_j\left(\varphi(t^n,x_j)-\varphi(t^{n-1},x_j)\right)\rho_j^n\\&+\lambda\sum_n\sum_j\left(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\right)g(\rho_j^n,\ldots,\rho_{j+N}^n)=0.\end{aligned}\tag{36}$$

Then we multiply (36) by Δx getting

$$\begin{aligned}&\Delta x\sum_j\varphi(0,x_j)\rho_j^0+\Delta x\Delta t\sum_n\sum_j\frac{\varphi(t^n,x_j)-\varphi(t^{n-1},x_j)}{\Delta t}\,\rho_j^n\\&+\Delta x\Delta t\sum_n\sum_j\frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\,g(\rho_j^n,\ldots,\rho_{j+N}^n)=0.\end{aligned}\tag{37}$$

By strong $\mathbf{L^1_{loc}}$ convergence of $\rho_{\Delta x}\to\rho$, it is straightforward to see that the first two terms in (37) converge to

$$\int_{-\infty}^{+\infty}\rho_0(x)\varphi(0,x)\,dx+\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho(t,x)\varphi_t(t,x)\,dxdt$$

as $\Delta x\searrow 0$. Concerning the last term, since $\rho_j^n\in[0,\rho_{\max}]$ and $|V_j^n|\leq v_{\max}$ for all $j\in\mathbb{Z}$, $n\in\mathbb{N}$, we observe that

$$\begin{aligned}&\left|g(\rho_j^n,\ldots,\rho_{j+N}^n)-\rho_j^nV_j^n\right|\\&\leq\frac{\alpha}{2}\left|\rho_{j+1}^n-\rho_j^n\right|+\frac{1}{2}\left|\rho_{j+1}^nV_{j+1}^n-\rho_j^nV_j^n\right|\\&\leq\frac{\alpha}{2}\left|\rho_{j+1}^n-\rho_j^n\right|+\frac{1}{2}\left|(\rho_{j+1}^n-\rho_j^n)V_{j+1}^n+\rho_j^n(V_{j+1}^n-V_j^n)\right|\end{aligned}$$

$$\leq \frac{v_{max} + \alpha}{2} \left|\rho^n_{j+1} - \rho^n_j\right| + \frac{1}{2}\,\rho^n_j \left|V^n_{j+1} - V^n_j\right|. \tag{38}$$

We rewrite $\left|V^n_{j+1} - V^n_j\right|$ using the mean value theorem

$$\begin{aligned}
\left|V^n_{j+1} - V^n_j\right| &= \left| v\left(\Delta x \sum_{k=0}^{N-1} w^k_\eta \rho^n_{j+k+1}\right) - v\left(\Delta x \sum_{k=0}^{N-1} w^k_\eta \rho^n_{j+k}\right)\right| \\
&= \left| v'(\xi)\Delta x \sum_{k=0}^{N-1} w^k_\eta (\rho^n_{j+k+1} - \rho^n_{j+k})\right| \\
&\leq -v'(\xi)\Delta x \sum_{k=0}^{N-1} w^k_\eta \left|\rho^n_{j+k+1} - \rho^n_{j+k}\right| \\
&\leq A\Delta x \sum_{k=0}^{N-1} w^k_\eta \left|\rho^n_{j+k+1} - \rho^n_{j+k}\right| \\
&\leq A\Delta x \; w_\eta(0)\mathrm{TV}(\rho_{\Delta x}(t^n,\cdot)),
\end{aligned}$$

therefore

$$\begin{aligned}
(38) &\leq \frac{v_{max} + \alpha}{2} \left|\rho^n_{j+1} - \rho^n_j\right| + \frac{1}{2} A \;\Delta x\; w_\eta(0)\mathrm{TV}(\rho_{\Delta x}(t^n, \dot{}))\\
&\leq \frac{v_{max} + \alpha}{2} \left|\rho^n_{j+1} - \rho^n_j\right| + C'(w_\eta, \rho_0, T)\;\Delta x,
\end{aligned} \tag{39}$$

where we have set $C'(w_\eta, \rho_0, T) = A\; w_\eta(0)C(w_\eta, \rho_0, T)/2$ for $T \geq t^n$, with $C(w_\eta, \rho_0, T)$ defined in (19).

Therefore, the last term in (37) can be rewritten as

$$\begin{aligned}
&\Delta x\Delta t \sum_n \sum_j \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x}\; g(\rho^n_j, \dots, \rho^n_{j+N}) \\
&= \Delta x\Delta t \sum_n \sum_j \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x}\; \rho^n_j V^n_j \\
&\quad + \Delta x\Delta t \sum_n \sum_j \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x}\left(g(\rho^n_j, \dots, \rho^n_{j+N}) - \rho^n_j V^n_j\right).
\end{aligned}$$

Again by $\mathbf{L^1_{loc}}$ convergence of $\rho_{\Delta x} \to \rho$ and boundedness of w_η, the first term in the above decomposition converges to

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \rho(t,x)v(\rho *_d w_\eta(t,x))\varphi_x(t,x)\; dxdt\;,$$

while the second term can be bounded using (39): set $T > 0$ and $R > 0$ such that $\varphi(t,x) = 0$ for $t > T$ and $|x| > R$, and let $n_T \in \mathbb{N}$ and $j_0, j_1 \in \mathbb{Z}$ such that $T \in \,]n_T\Delta t, (n_T+1)\Delta t]$, $-R \in \,]x_{j_0-1/2}, x_{j_0+1/2}]$ and $R \in \,]x_{j_1-1/2}, x_{j_1+1/2}]$, then

$$\begin{aligned}
&\Delta x\Delta t \sum_n \sum_j \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x}\;\left(g(\rho^n_j, \dots, \rho^n_{j+N}) - \rho^n_j V^n_j\right) \\
&\leq \Delta x\Delta t\|\varphi_x\|_\infty \sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\left(\frac{v_{max}+\alpha}{2}\left|\rho^n_{j+1} - \rho^n_j\right| + C'(w_\eta, \rho_0, T)\;\Delta x\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{v_{max}+\alpha}{2}\|\varphi_x\|_\infty \Delta x\Delta t\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\left|\rho_{j+1}^n-\rho_j^n\right| + \|\varphi_x\|_\infty C'(w_\eta,\rho_0,T)2RT\Delta x\\
&\leq \frac{v_{max}+\alpha}{2}\|\varphi_x\|_\infty \int_0^T\int_{-R}^R |\rho_{\Delta x}(t,x+\Delta x)-\rho_{\Delta x}(t,x)|\, dxdt\\
&\quad + \|\varphi_x\|_\infty C'(w_\eta,\rho_0,T)2RT\Delta x\\
&\leq \frac{v_{max}+\alpha}{2}\|\varphi_x\|_\infty C(w_\eta,\rho_0,T)\Delta x + \|\varphi_x\|_\infty C'(w_\eta,\rho_0,T)2RT\Delta x\ ,
\end{aligned}
$$

which clearly goes to zero when $\Delta x \searrow 0$.

Concerning the entropy condition, we proceed as above to show that (32) converges to (5). Multiplying (32) by $\Delta x\, \varphi(t^n,x_j)\geq 0$ and then summing by parts we get

$$
\begin{aligned}
0 \leq\ & \Delta x\sum_j \varphi(0,x_j)\left|\rho_j^0-\kappa\right| + \Delta x\Delta t\sum_n\sum_j \frac{\varphi(t^n,x_j)-\varphi(t^{n-1},x_j)}{\Delta t}\ \left|\rho_j^n-\kappa\right|\\
&+ \Delta x\Delta t\sum_n\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\ F_{j+1/2}^\kappa(\rho_j^n,\rho_{j+1}^n)\\
&- \Delta x\Delta t\sum_n\sum_j \operatorname{sgn}\left(\rho_j^{n+1}-\kappa\right)\kappa\ \frac{V_{j+1}^n-V_{j-1}^n}{2\Delta x}\ \varphi(t^n,x_j)\ .
\end{aligned}
$$

Following the same steps as above, the first three terms in the sum clearly converge to

$$
\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(|\rho(t,x)-\kappa|\varphi_t + |\rho(t,x)-\kappa| v(\rho *_d w_\eta(t,x))\ \varphi_x\right) dxdt\\
&\quad + \int_{-\infty}^{+\infty} |\rho_0(x)-\kappa|\varphi(0,x)\ dx
\end{aligned}
$$

as $\Delta x\searrow 0$. The third term can be decomposed as

$$
\begin{aligned}
&\sum_n\sum_j \operatorname{sgn}\left(\rho_j^{n+1}-\kappa\right)\kappa\ \frac{V_{j+1}^n-V_{j-1}^n}{2\Delta x}\ \varphi(t^n,x_j)\\
&= \left(\operatorname{sgn}\left(\rho_j^{n+1}-\kappa\right)-\operatorname{sgn}\left(\rho_j^n-\kappa\right)\right)\kappa\ \frac{V_{j+1}^n-V_{j-1}^n}{2\Delta x}\ \varphi(t^n,x_j)\\
&\quad + \operatorname{sgn}\left(\rho_j^n-\kappa\right)\kappa\ \frac{V_{j+1}^n-V_{j-1}^n}{2\Delta x}\ \varphi(t^n,x_j)\ .
\end{aligned}
\tag{40}
$$

The first term in (40) can be controlled by $C(w_\eta,\rho_0)\Delta x$, and the second clearly converges to

$$
-\int_0^{+\infty}\int_{-\infty}^{+\infty}\operatorname{sgn}(\rho(t,x)-\kappa)\,\kappa\, V_x(t,x)\ \varphi(t,x)dxdt\ ,
$$

providing the entropy inequality (5).

2.6 Lipschitz continuity of the solution

Following [3], we prove that if the initial datum is Lipschitz continuous, the numerical solution converges to a Lipschitz continuous function.

Lemma 2. *For any $T > 0$, the numerical solution generated by the scheme* (10), (11) *converges to a Lipschitz continuous function ρ, provided ρ_0 is also Lipschitz continuous.*

Proof. Defining $\Omega_{j+k-\frac{1}{2}} = \Delta_{j+k-\frac{1}{2}}/\Delta x$ we obtain from (25)

$$\begin{aligned}
\Omega^{n+1}_{j+\frac{1}{2}} = \frac{\lambda}{2}&\left[\alpha + V_j + \rho_{j-1}\Delta x\ v'(\xi)w^0_\eta - \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w^{k-1}_\eta \Delta_{j+k+\frac{1}{2}}\right]\Omega_{j-\frac{1}{2}}\\
&+\left[1-\lambda\alpha + \frac{\lambda}{2}\rho_{j-1}\Delta x\ v'(\xi)w^1_\eta - \frac{\lambda}{2}\Delta x\ v'(\xi')\sum_{k=2}^{N-2} w^{k-1}_\eta\Delta_{j+k+\frac{1}{2}}\right]\Omega_{j+\frac{1}{2}}\\
&+\frac{\lambda}{2}\left[\alpha - V_{j+2} + \rho_{j-1}\Delta x\ v'(\xi)w^2_\eta - \rho_{j+1}\Delta x\ v'(\xi')w^0_\eta\right]\Omega_{j+\frac{3}{2}}\\
&+\frac{\lambda}{2}\Delta x\ \sum_{k=2}^{N-2}\Omega_{j+k+\frac{1}{2}}\left[\rho_{j-1}\ v'(\xi)(w^{k+1}_\eta - w^{k-1}_\eta) + w^{k-1}_\eta\rho_{j-1}\left(v'(\xi)-v'(\xi')\right)\right]\\
&-\frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w^{N-2}_\eta\Omega_{j+N-\frac{1}{2}}\\
&-\frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w^{N-1}_\eta\Omega_{j+N+\frac{1}{2}}.
\end{aligned}$$

Taking the absolute value in the above expression we get

$$\begin{aligned}
\left|\Omega^{n+1}_{j+\frac{1}{2}}\right| \leq \|\Omega^n\|_\infty\Bigg[1+\frac{\lambda}{2}\Bigg(&\alpha + V_j + \rho_{j-1}\Delta x\ v'(\xi)w^0_\eta - \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w^{k-1}_\eta\Delta_{j+k+\frac{1}{2}}\\
&-2\alpha + \rho_{j-1}\Delta x\ v'(\xi)w^1_\eta - \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w^{k-1}_\eta\Delta_{j+k+\frac{1}{2}}\\
&+\alpha - V_{j+2} + \rho_{j-1}\Delta x\ v'(\xi)w^2_\eta - \rho_{j+1}\Delta x\ v'(\xi')w^0_\eta\\
&+\Delta x\ \sum_{k=2}^{N-2}\left|\rho_{j-1}\ v'(\xi)(w^{k+1}_\eta - w^{k-1}_\eta) + w^{k-1}_\eta\rho_{j-1}(v'(\xi)-v'(\xi'))\right|\\
&-\rho_{j+1}\ \Delta x\ v'(\xi')\ w^{N-2}_\eta\\
&-\rho_{j+1}\ \Delta x\ v'(\xi')\ w^{N-1}_\eta\Bigg)\Bigg].
\end{aligned}\tag{41}$$

Observing that

$$V_j - V_{j+2} = v'(\bar\xi)\Delta x\ \sum_{k=0}^{N-1} w^k_\eta(\rho_{j+k} - \rho_{j+k+2}),$$

with $\bar\xi$ between $\Delta x\ \sum_{k=0}^{N-1} w^k_\eta\rho_{j+k}$ and $\Delta x\ \sum_{k=0}^{N-1} w^k_\eta\rho_{j+k+2}$, we have that the coefficient on the right-han d side of (41) is bounded by

$$\begin{aligned}
1+\frac{\Delta t}{2}\Bigg[&\rho_{j-1}v'(\xi)(w^0_\eta + w^1_\eta + w^2_\eta)\\
&-2\sum_{k=2}^{N-2}\Delta_{j+k+\frac{1}{2}}w^{k-1}_\eta v'(\xi') &\text{(42a)}\\
&-v'(\bar\xi)\sum_{k=0}^{N-1} w^k_\eta(\rho_{j+k+2} - \rho_{j+k}) &\text{(42b)}\\
&-\rho_{j+1}v'(\xi')(w^0_\eta + w^{N-2}_\eta + w^{N-1}_\eta)
\end{aligned}$$

$$
\begin{aligned}
&+\rho_{j-1}\sum_{k=2}^{N-2} v'(\xi)(w_\eta^{k+1}-w_\eta^{k-1})\\
&+\sum_{k=2}^{N-2}\rho_{j-1}w_\eta^{k-1}\underbrace{|v'(\xi)-v'(\xi')|}_{=\|v''\|_\infty 7w_\eta(0)\Delta x}\Bigg].
\end{aligned}
$$

We rewrite

$$
\begin{aligned}
(42\text{a}) &= -2v'(\xi')\sum_{k=2}^{N-2} w_\eta^{k-1}(\rho_{j+k+1}-\rho_{j+k})\\
&= -2v'(\xi')\left[\sum_{k=3}^{N-1} w_\eta^{k-2}\rho_{j+k}-\sum_{k=2}^{N-2} w_\eta^{k-1}\rho_{j+k}\right]\\
&= -2v'(\xi')\left[\sum_{k=3}^{N-2}(w_\eta^{k-2}-w_\eta^{k-1})\rho_{j+k}+w_\eta^{N-3}\rho_{j+N-1}-w_\eta^{1}\rho_{j+2}\right]
\end{aligned}
$$

$$
\begin{aligned}
(42\text{b}) &= -v'(\bar\xi)\left[\sum_{k=2}^{N+1} w_\eta^{k-2}\rho_{j+k}-\sum_{k=0}^{N-1} w_\eta^{k}\rho_{j+k}\right]\\
&= -v'(\bar\xi)\Big[\sum_{k=2}^{N-1}(w_\eta^{k-2}-w_\eta^{k})\rho_{j+k}+w_\eta^{N-2}\rho_{j+N}+w_\eta^{N-1}\rho_{j+N+1}-w_\eta^{0}\rho_{j}-w_\eta^{1}\rho_{j+1}\Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(42) \le 1+\frac{\Delta t}{2}\Bigg[&2A\underbrace{\sum_{k=3}^{N-2}(w_\eta^{k-2}-w_\eta^{k-1})}_{=w_\eta^1-w_\eta^{N-3}}+2Aw_\eta^{N-3}\\
&+A\underbrace{\sum_{k=2}^{N-1}(w_\eta^{k-2}-w_\eta^{k})}_{=w_\eta^0+w_\eta^1-w_\eta^{N-2}-w_\eta^{N-1}}+Aw_\eta^{N-2}+Aw_\eta^{N-1}+A(w_\eta^0+w_\eta^{N-2}+w_\eta^{N-1})\\
&+A\underbrace{\sum_{k=2}^{N-2}(w_\eta^{k-1}-w_\eta^{k+1})}_{=w_\eta^1+w_\eta^2-w_\eta^{N-2}-w_\eta^{N-1}}+7\|v''\|_\infty w_\eta(0)\underbrace{\Delta x\sum_{k=2}^{N-2}w_\eta^{k-1}}_{\le 1}\Bigg]\\
&\le 1+\frac{\Delta t}{2}\left(7w_\eta(0)A+7w_\eta(0)\|v''\|_\infty\right)\\
&\le 1+7\frac{\Delta t}{2}w_\eta(0)\left(A+\|v''\|_\infty\right)
\end{aligned}
$$

which substituted in (41) gives

$$
\begin{aligned}
\left|\Omega_{j+\frac12}^{n+1}\right| &\le \|\Omega^n\|_\infty\left[1+7\frac{\Delta t}{2}w_\eta(0)\left(A+\|v''\|_\infty\right)\right]\\
&\le \left\|\Omega^0\right\|_\infty\left[1+7\frac{\Delta t}{2}w_\eta(0)\left(A+\|v''\|_\infty\right)\right]^{\frac{T}{\Delta t}} \le \left\|\Omega^0\right\|_\infty e^{7\frac{T}{2}w_\eta(0)(A+\|v''\|_\infty)}.
\end{aligned}
$$

To conclude we note that

$$\Omega^0_{j+\frac{1}{2}} = \frac{\rho^0_{j+1}-\rho^0_j}{\Delta x} = \frac{1}{\Delta x^2}\left(\int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{3}{2}}} \rho_0(y)dy - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho_0(y)dy\right)$$
$$= \frac{1}{\Delta x^2}\left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\rho_0(y+\Delta x)-\rho_0(y)\right)dy\right) \leq \|\rho_0\|_{\mathrm{Lip}}.$$

For the discrete time derivative, using the scheme (10), (11) we can write

$$\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t} = \frac{\lambda\alpha}{2}\left(\frac{\rho^n_{j+1}-\rho^n_j}{\Delta t} - \frac{\rho^n_j-\rho^n_{j-1}}{\Delta t}\right) - \frac{\lambda}{2}\frac{V^n_{j+1}}{\Delta t}(\rho^n_{j+1}-\rho^n_{j-1}) - \frac{\lambda}{2}\frac{\rho^n_{j-1}}{\Delta t}(V^n_{j+1}-V^n_{j-1})$$
$$= \alpha\left(\frac{\rho^n_{j+1}-\rho^n_j}{2\Delta x} - \frac{\rho^n_j-\rho^n_{j-1}}{2\Delta x}\right) - \frac{V^n_{j+1}}{2\Delta x}(\rho^n_{j+1}-\rho^n_{j-1}) - \frac{\rho^n_{j-1}}{2\Delta x}(V^n_{j+1}-V^n_{j-1}).$$

Taking the absolute value we get

$$\left|\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}\right| \leq \frac{\alpha}{2}\left(\frac{\left|\rho^n_{j+1}-\rho^n_j\right|}{\Delta x} + \frac{\left|\rho^n_j-\rho^n_{j-1}\right|}{\Delta x}\right) + \frac{V^n_{j+1}}{2\Delta x}\left|\rho^n_{j+1}-\rho^n_{j-1}\right| + \frac{\rho^n_{j-1}}{2\Delta x}\left|V^n_{j+1}-V^n_{j-1}\right|.$$

By the mean value theorem we have

$$\frac{\left|V^n_{j+1}-V^n_{j-1}\right|}{2\Delta x} = \left|v'(\hat{\xi})\Delta x \sum_{k=0}^{N-1} w^k_\eta \frac{\rho^n_{j+k+1}-\rho^n_{j+k-1}}{2\Delta x}\right| \leq A(1+w_\eta(0)\Delta x)\|\Omega^n\|_\infty.$$

Then, using that $\rho_{\Delta x}(t,x)$ is Lipschitz continuous respect to the space variable:

$$\left|\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}\right| \leq \frac{\alpha}{2}\cdot 2\ \mathrm{Lip}_x(\rho_{\Delta x}) + v_{\max}\ \mathrm{Lip}_x(\rho_{\Delta x}) + A\left(1+w_\eta(0)\Delta x\right)\|\Omega^n\|_\infty$$
$$= \left[\alpha + v_{max} + A\left(1+w_\eta(0)\Delta x\right)\right]\|\Omega^n\|_\infty,$$

we get that the solution generated by the numerical method converges to a Lipschitz continuous function. □

2.7 Numerical Results

In this section, we perform numerical simulations to show evidence of some properties of equation (1). In particular, we compare solutions depending on the choices of the kernel and the velocity functions. More precisely, we will use the following velocity functions, see in [9]:

Greenshield: $v(\rho) = v_{max}\left(1-\left(\frac{\rho}{\rho_{max}}\right)^n\right), \quad n\in\mathbb{N},$

Greenberg: $v(\rho) = v_{max}\log\left(\frac{\rho_{max}}{\rho}\right),$

Underwood: $v(\rho) = v_{max}\exp\left(-\frac{\rho}{\rho_{max}}\right),$

California: $v(\rho) = v_{max}\left(\frac{1}{\rho}-\frac{1}{\rho_{max}}\right),$

with $v_{max} = 1$ and $\rho_{max} = 1$, and the kernels $w_\eta \in \mathbf{C^1}([0,\eta];\mathbb{R}^+)$:

$$\begin{aligned}
&\text{constant:} && w_\eta(x) = \frac{1}{\eta},\\
&\text{linear decreasing:} && w_\eta(x) = 2\frac{\eta - x}{\eta^2},\\
&\text{convex decreasing:} && w_\eta(x) = 3\frac{(\eta - x)^2}{\eta^3},\\
&\text{concave decreasing:} && w_\eta(x) = 3\frac{\eta^2 - x^2}{2\eta^3},\\
&\text{linear increasing:} && w_\eta(x) = \frac{2x}{\eta^2},
\end{aligned}$$

Remark that in [4] the authors considered the Greenshield model with $n = 1$ (i.e. linear velocity).

For the test presented below, the space domain is given by the interval $[-1, 1]$, the space discretization mesh $\Delta x = 0.002$ and $\eta = 0.1$, where not specified otherwise. Absorbing conditions are imposed at the boundaries. More precisely, at the right boundary: we add $N = \eta/\Delta x$ ghost cells and define $\rho_j^n = \rho^n_{\frac{2}{\Delta x}}$ for every $j = \frac{2}{\Delta x}+1, \ldots, \frac{2}{\Delta x}+N$, thus extending the solution constantly equal to the last value inside the domain. This choice is particularly suitable for computing the solutions of Riemann problems. At the left boundary, we just need to add one ghost cell, as in classical problems. We compute solutions at different times to show clearly the properties.

We compute the solution of the Riemann problem with initial datum

$$\rho_0(x) = \begin{cases} \rho_L, & \text{if } x < 0,\\ \rho_R, & \text{if } x > 0, \end{cases} \tag{43}$$

where $\rho_L = 0.2$ and $\rho_R = 0.8$ if not specified otherwise.

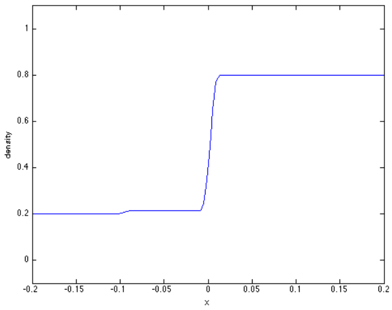
2.7.1 Kernel monotonicity

We are interested in studying the effect of the monotonicity of the kernel on the solution characteristics.

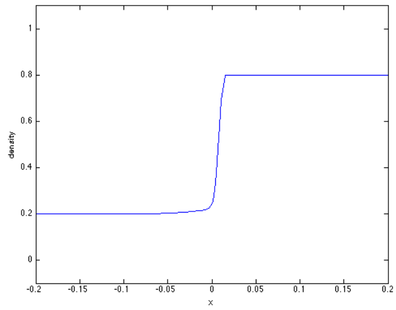
The table below describe the Total Variation (if it is constant or not) and the monotonicity preservation of density ρ, with Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$:

	w_η constant		w_η lin decr		w_η convex		w_η concave		w_η increasing	
	TV	MP	TV	MP	TV	MP	TV	MP	TV	MP
Greenshield $n = 1$	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗
Greenshield $n = 5$	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗
Greenberg	✗	✗	✓	✓	✓	✓	✗	✗	✗	✗
Underwood	✗	✗	✓	✓	✓	✓	✗	✗	✗	✗
California	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗

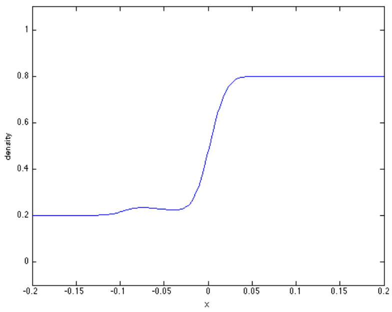
In Figure 1 we compare the solutions corresponding to the constant kernel $w_\eta = 1/\eta$. Numerical simulations show that the scheme generally is not monotonicity preserving, as shown in the case of the Greenberg's and the Underwood's velocities, and the related total variations (Figure 2) are not constant, instead Greenshield's velocities compute a monotonicity preserving solution. For a linear increasing kernel, all the computed solutions are not monotonicity preserving and



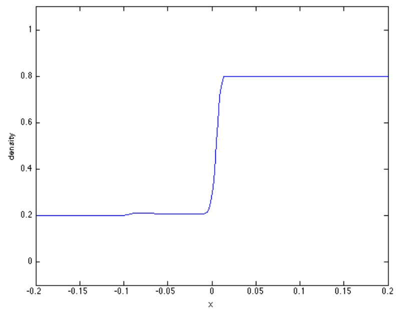
(a) Greenshield's velocity, $n = 1$



(b) Greenshield's velocity, $n = 5$



(c) Greenberg's velocity



(d) Underwood's velocity

Figure 1: Density profiles that solve the equation (1) at time $t = 0.01$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$, $\rho_R = 0.8$ and kernel $w_\eta(x) = 1/\eta$. Cases (c) (d) show that the corresponding numerical scheme is not monotonicity preserving.

the total variation increases for all velocity function as shown in Figure3 for the linear velocity function.

We also note that the kernel monotonicity influences the solution monotonicity (Figure 4), indeed the solution obtained with Underwood's velocity is monotone if the kernel is linear decreasing or convex.

2.7.2 Classical equation

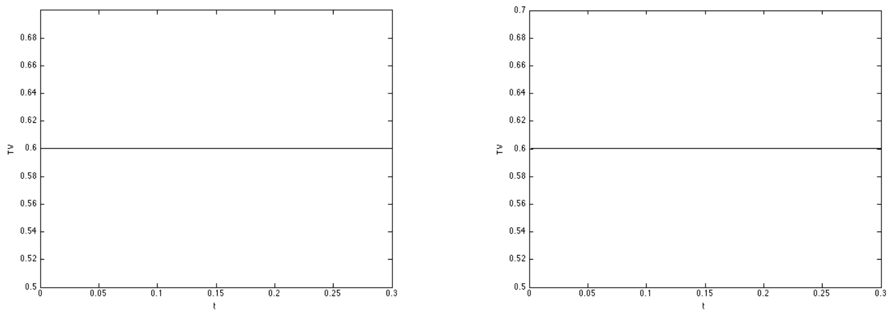
In this section we solve the classical equation

$$\partial_t \rho + \partial_x(\rho v(\rho)) = 0,$$

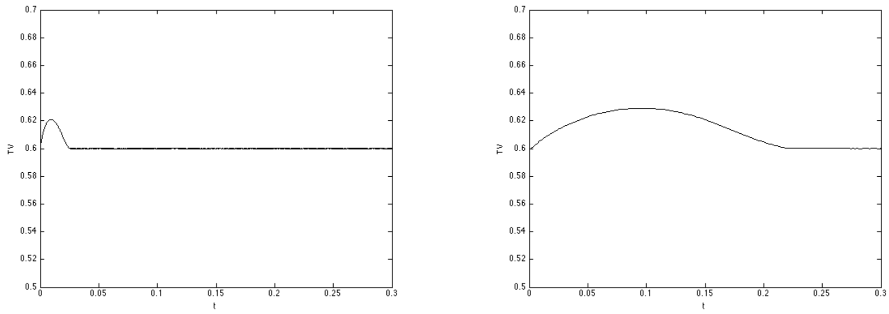
with initial condition

$$\rho_0(x) = \begin{cases} 0.2 & \text{if } x < 0 \\ 0.8 & \text{if } x > 0 \end{cases}.$$

We have a different shock profile for each kind of velocity like in Figure 5.



(a) Greenshield's velocity, $n = 1$ (b) Greenshield's velocity, $n = 5$



(c) Greenberg's velocity (d) Underwood's velocity

Figure 2: Total Variation $\mathrm{TV}(\rho(t,\cdot);[-1,1])$ for $t \in [0, 0.3]$ corresponding to the Riemann-like initial datum with $\rho_L = 0.2$, $\rho_R = 0.8$ for $w(x) = 1/\eta$.

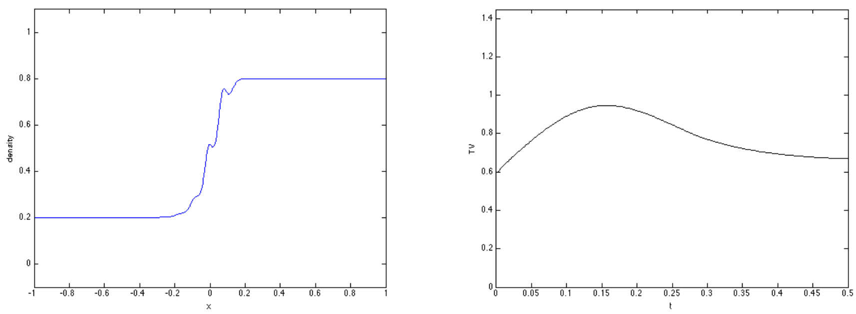
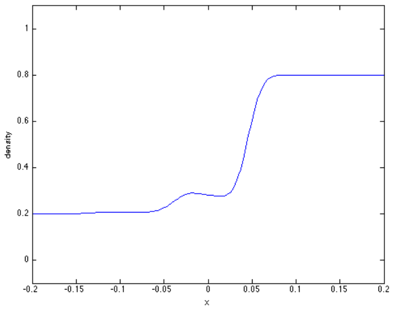
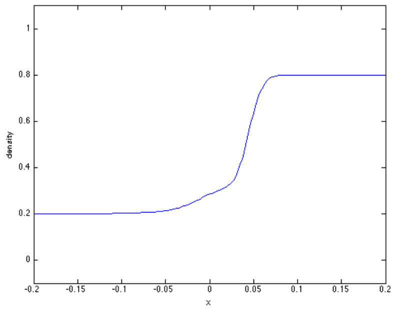


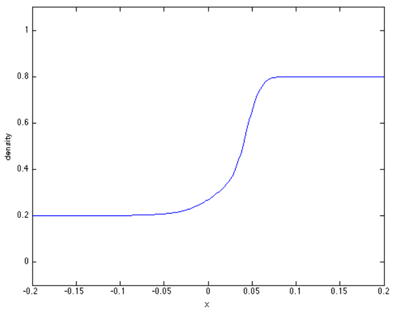
Figure 3: Density profile at $t = 0.5$ (left) and the total variation $\mathrm{TV}(\rho(t,\cdot);[-1,1])$ for $t \in [0, 0.5]$ (right) corresponding to the Riemann-like initial datum with $\rho_L = 0.2$, $\rho_R = 0.8$ for $w_\eta(x) = 2x/\eta^2$ and linear velocity.



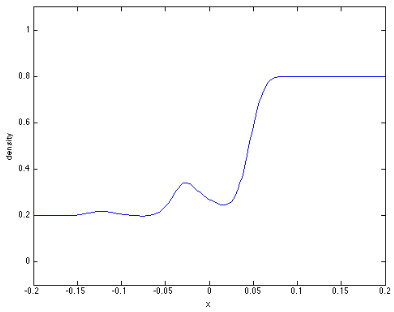
(a) $w_\eta = 1/\eta$ constant



(b) $w_\eta = 2(\eta - x)/\eta^2$ linear

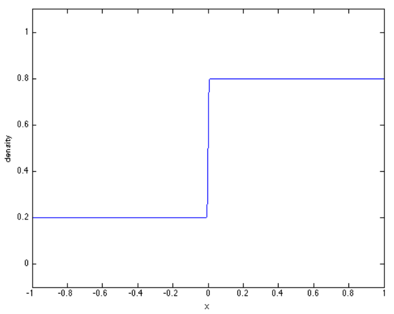


(c) $w_\eta = 3(\eta - x)^2/\eta^3$ convex

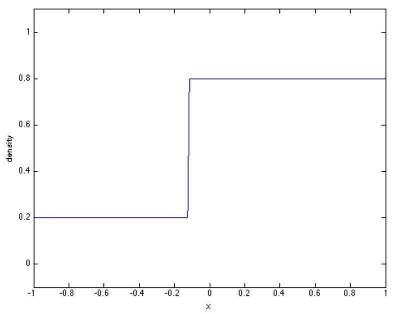


(d) $w_\eta = 2x/\eta^2$ increasing

Figure 4: Density profiles at time $t = 0.1$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$, $\rho_R = 0.8$ with Underwood's velocity with different kernels to show that the scheme is not monotonicity preserving.



(a) linear velocity



(b) Greenberg's velocity

Figure 5: Density profiles at $t = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$, $\rho_R = 0.8$ that solves the classical equation.

Remark. *Concerning the CFL condition, we want*

$$\Delta t < \frac{\Delta x}{\lambda_{max}} \quad \text{with } \lambda_{max} = max_{\rho \in [\rho_m, \rho_M]} |f'(\rho)|,$$

where ρ_m and ρ_M are defined in (15) *and $f'(\rho)$ is the first derivative of the flux $f(\rho) = \rho v(\rho)$. Of course, the CFL condition depends on the choice of the velocity function. For example, in the case of Greenshield's velocity $v = v_{max}\left(1 - \left(\frac{\rho}{\rho_{max}}\right)^n\right)$ we have $\lambda_{max} = n v_{max}$, indeed:*

$$f'(\rho) = v(\rho) + \rho v'(\rho), \qquad v'(\rho) = -n \frac{v_{max}}{\rho_{max}} \left(\frac{\rho}{\rho_{max}}\right)^{n-1}$$

$$\begin{aligned} f'(\rho) &= v_{max}\left(1 - \left(\frac{\rho}{\rho_{max}}\right)^n\right) - n\rho \frac{v_{max}}{\rho_{max}} \left(\frac{\rho}{\rho_{max}}\right)^{n-1} \\ &= v_{max} - v_{max}\left(\frac{\rho}{\rho_{max}}\right)^n - n v_{max}\left(\frac{\rho}{\rho_{max}}\right)^n \\ &= v_{max}\left(1 - (1+n)\left(\frac{\rho}{\rho_{max}}\right)^n\right), \end{aligned}$$

so $\lambda_{max} = max_{\rho \in (\rho_m, \rho_M)} |f'(\rho)| \leq n v_{max}$.

2.7.3 Convergence orders

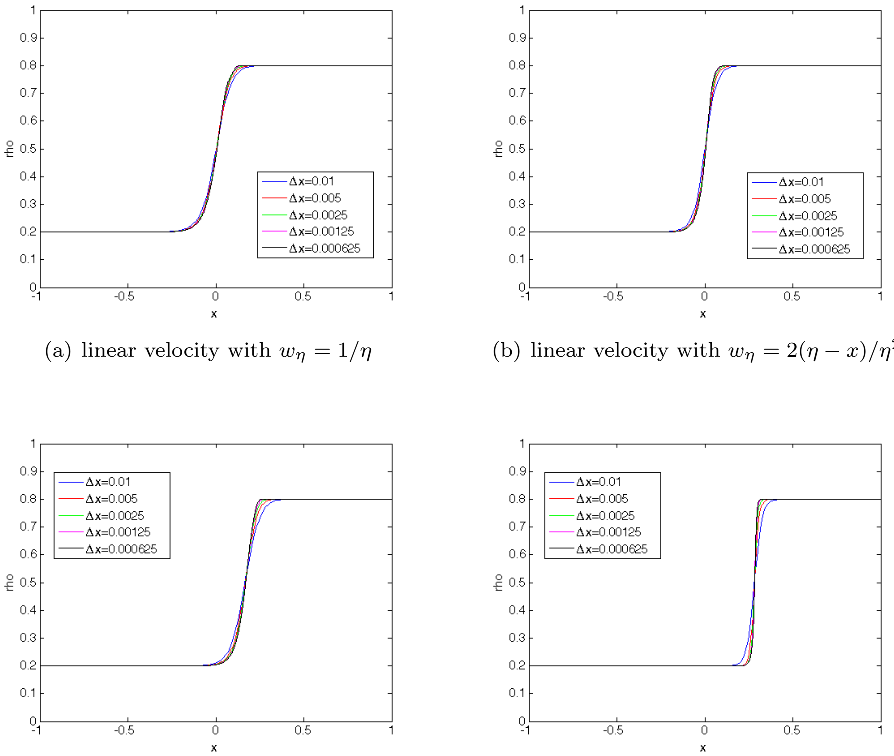
In this Section we relate the numerical convergence orders for the scheme (10), (11) comparing the solution where Δx is halving successively (Figure 6). Halving the parameter Δx and looking at the ratios of the errors $\rho_{\Delta x} - \rho_{\frac{\Delta x}{2}}$ and $\rho_{\frac{\Delta x}{2}} - \rho_{\frac{\Delta x}{4}}$ we define

$$\gamma(\Delta x) = \log_2 \left(\frac{e(\Delta x)}{e(\Delta x/2)}\right), \tag{44}$$

see [5], where the $\mathbf{L^1}$-error is computed at final time $T = 0.5$ as

$$e(\Delta x) = \left\| \rho_{\Delta x}(T, \cdot) - \rho_{\frac{\Delta x}{2}}(T, \cdot) \right\|_{\mathbf{L^1}} = \frac{\Delta x}{2} \sum_j \left| \rho_{\Delta x}(T, x_j) - \rho_{\frac{\Delta x}{2}}(T, x_j) \right|, \tag{45}$$

for $\Delta x = 0.01, 0.005, 0.0025, 0.00125, 0.000625$. The tables below show the $\mathbf{L^1}$-error and the convergence order of the numerical solution with Riemann-like initial datum with $\rho_L = 0.2$, $\rho_R = 0.8$ and $\eta = 0.1$. We show how the kernel monotonicity influences the convergence order. We consider the linear decreasing velocity $v(\rho) = 1 - \rho$, the Underwood's velocity $v(\rho) = e^{-\rho}$ and the Greenshield's velocity $v(\rho) = 1 - \rho^5$. We note that in case of linear increasing kernel, convergence is not clearly established.



(a) linear velocity with $w_\eta = 1/\eta$

(b) linear velocity with $w_\eta = 2(\eta - x)/\eta^2$

(c) Underwood's velocity with $w_\eta = 2(\eta - x)/\eta^2$

(d) Greenshield's velocity ($n = 5$) with $w_\eta = 2(\eta - x)/\eta^2$

Figure 6: Density profiles that solve the equation (1) at time $t = 0.5$ with different space steps $\Delta x = 0.01, 0.005, 0.0025, 0.00125, 0.000625$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$, $\rho_R = 0.8$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$		$w_\eta(x) = 2x/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.996250	4.225405 e-03	1.045449	4.904882 e-03	-0.318483	3.778952 e-03
0.005	0.985828	2.118200 e-03	1.018527	2.376385 e-03	-0.571840	4.712426 e-03
0.0025	0.970396	1.069555 e-03	1.001553	1.173031 e-03	-0.033678	7.004638 e-03
0.00125	0.701916	5.458643 e-04	1.006433	5.858843 e-04	0.095021	7.170079 e-03
0.000625	0.616415	3.355728 e-04	1.001958	2.916388 e-04	0.258423	6.713045 e-03

Table 1: Convergence orders and $\mathbf{L^1}$-errors for constant, linear decreasing and linear increasing kernels with linear decreasing velocity $v(\rho) = 1 - \rho$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$		$w_\eta(x) = 2x/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.932941	5.446250 e-03	0.992245	6.490031 e-03	-0.191595	4.783854 e-03
0.005	0.626900	2.852687 e-03	0.955618	3.262505 e-03	-0.315515	5.463287 e-03
0.0025	0.344915	1.847304 e-03	1.065079	1.682214 e-03	0.109477	6.798823 e-03
0.00125	0.403401	1.454482 e-03	0.959188	8.040085 e-04	0.403449	6.301988 e-03
0.000625	0.496276	1.099695 e-03	0.772357	4.135386 e-04	0.533528	4.764607 e-03

Table 2: Convergence orders and $\mathbf{L^1}$-errors for constant, linear decreasing and linear increasing kernels with Underwood's velocity $v(\rho) = e^{-\rho}$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$		$w_\eta(x) = 2x/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	1.205060	5.580313 e-03	1.332522	7.630177 e-03	0.358133	2.953069 e-03
0.005	0.987451	2.420468 e-03	1.208013	3.029740 e-03	0.174693	2.303905 e-03
0.0025	1.313340	1.220806 e-03	1.037360	1.311466 e-03	0.016130	2.041158 e-03
0.00125	0.937723	4.912381 e-04	1.196083	6.389704 e-04	-0.008435	2.018463 e-03
0.000625	1.028282	2.564538 e-04	1.141780	2.788841 e-04	0.140867	2.030300 e-03

Table 3: Convergence orders and $\mathbf{L^1}$-errors for constant and linear decreasing and linear increasing kernels with Greenshield's velocity $v(\rho) = 1 - \rho^5$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$.

3 Second order scheme

In this Section, we develop a second order central scheme for (1) inspired by the one in [11], which focused on a traffic flow model with Arrhenius look-ahead dynamics (see [2, 15, 17]). The proposed scheme is an extension of the first-order staggered Lax-Friedrichs scheme and the second-order Nessyahu-Tadmor scheme [14], which belong to a class of Godunov-type methods. A solution, computed at a certain time, is first approximated by a piecewise linear function instead of a piecewise-constant one, which is then evolved to the next time step according to the integral form of conservation law.

If we define the convolution term as:

$$R(t,x) = \int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\ dy \tag{46}$$

and the flux is

$$F(\rho, R) = \rho v(R) = \rho(t,x) v\Big(\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\ dy \Big),$$

we rewrite (1) as

$$\partial_t \rho(t,x) + \partial_x F(\rho, R) = 0, \tag{47}$$

defined for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, with initial datum $\rho(0,x) = \rho_0(x) \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$

3.1 Derivation of the central scheme

We introduce the following notation: $x_j = x_{\min} + (j - 1/2)\Delta x$, $t^n = n\Delta t$, where Δx and Δt are space and time step. The computational space domain $[x_{\min}, x_{\max}]$ is divided into cells $C_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ so that

$$x_{\max} = x_{\min} + K\Delta x \quad \text{and} \quad [x_{\min}, x_{\max}] = \bigcup_{j=1}^{K} C_j. \tag{48}$$

Let us assume that at time $t = t^n$ we have the solution realized by its cell averages

$$\bar{\rho}_j^n := \frac{1}{\Delta x} \int_{C_j} \rho(x, t^n)\ dx$$

then we construct its piecewise linear interpolant

$$\tilde{\rho}^n(x) := \bar{\rho}_j^n + s_j^n (x - x_j), \qquad x \in C_j. \tag{49}$$

Formally we have a second-order approximation provided the slopes s_j^n are at least first-order approximations of the derivatives $\rho_x(t^n, x_j)$. We use a generalized minmod reconstruction (see [12, 13, 14, 16]) with:

$$s_j^n = \operatorname{minmod}\left(\theta \frac{\bar{\rho}_j^n - \bar{\rho}_{j-1}^n}{\Delta x}, \frac{\bar{\rho}_{j+1}^n - \bar{\rho}_{j-1}^n}{2\Delta x}, \theta \frac{\bar{\rho}_{j+1}^n - \bar{\rho}_j^n}{\Delta x} \right), \quad \theta \in [1,2], \tag{50}$$

where the minmod function is defined as:

$$\operatorname{minmod}(z_1, z_2, \dots) := \begin{cases} \min_j\{z_j\}, & \text{if } z_j > 0 \quad \forall j, \\ \max_j\{z_j\}, & \text{if } z_j < 0 \quad \forall j, \\ 0, & \text{otherwise,} \end{cases}$$

and the parameter θ is used to control the numerical viscosity of the scheme: larger values of θ correspond to less dissipative but more oscillatory approximations. We now evolve the approximated solution (49) to the next time level $t = t^{n+1}$ by integrating equation (47) over the space-time volumes $[x_j, x_{j-1}] \times [t^n, t^{n+1}]$, which gives:

$$\begin{aligned}\bar{\rho}^{n+1}_{j+\frac{1}{2}} = & \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \tilde{\rho}^n(x)dx \\ & - \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \left[F(\rho(t,x_{j+1}), R(t,x_{j+1})) - F(\rho(t,x_j), R(t,x_j))\right] dt. \end{aligned} \tag{51}$$

The first integral of the right-hand side of (51) can be computed exactly:

$$\begin{aligned}\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \tilde{\rho}^n(x)dx &= \frac{1}{\Delta x}\int_{x_j}^{x_{j+\frac{1}{2}}} \tilde{\rho}^n(x)dx + \frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} \tilde{\rho}^n(x)dx \\ &= \frac{1}{\Delta x}\int_{x_j}^{x_{j+\frac{1}{2}}} [\bar{\rho}^n_j + s^n_j(x-x_j)]dx + \frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} [\bar{\rho}^n_{j+1} + s^n_{j+1}(x-x_{j+1})]dx \\ &= \frac{1}{\Delta x}\left[\frac{\Delta x}{2}\bar{\rho}^n_j + s^n_j\frac{(x-x_j)^2}{2}\Bigg|_{x_j}^{x_{j+\frac{1}{2}}} + \frac{\Delta x}{2}\bar{\rho}^n_{j+1} + s^n_{j+1}\frac{(x-x_{j+1})^2}{2}\Bigg|_{x_{j+\frac{1}{2}}}^{x_{j+1}}\right] \\ &= \frac{\bar{\rho}^n_j}{2} + \frac{s^n_j}{2\Delta x}\left(\frac{\Delta x}{2}\right)^2 + \frac{\bar{\rho}^n_{j+1}}{2} - \frac{s^n_{j+1}}{2\Delta x}\left(\frac{\Delta x}{2}\right)^2 \\ &= \frac{\bar{\rho}^n_j + \bar{\rho}^n_{j+1}}{2} + \frac{\Delta x}{8}(s^n_j - s^n_{j+1}).\end{aligned}$$

The flux integral in (51) should be computed using the approximate solution of the initial value problem (47), (49) in the time interval (t^n, t^{n+1}) with initial data obtained at $t = t^n$. Due to the finite propagation speed, we introduce the following CFL condition

$$\Delta t < \frac{1}{2}\frac{\Delta x}{\lambda_{\max}} \quad \text{with } \lambda_{\max} := \max_{\rho\in[\rho_{\mathrm{m}},\rho_{\mathrm{M}}]}\left|\frac{d}{d\rho}\rho v(\rho)\right|.$$

Therefore, we can compute the flux integrals in (51) by the mid-point quadrature:

$$\begin{aligned}\bar{\rho}^{n+1}_{j+\frac{1}{2}} =& \frac{\bar{\rho}^n_j + \bar{\rho}^n_{j+1}}{2} + \frac{\Delta x}{8}(s^n_j - s^n_{j+1}) \\ & - \lambda\left[F(\rho(t^{n+\frac{1}{2}}, x_{j+1}), R(t^{n+\frac{1}{2}}, x_{j+1})) - F(\rho(t^{n+\frac{1}{2}}, x_j), R(t^{n+\frac{1}{2}}, x_j))\right],\end{aligned} \tag{52}$$

where we approximate the intermediate time level values of ρ and R with the corresponding Taylor expansions, namely:

$$\rho(t^{n+\frac{1}{2}}, x_j) \approx \tilde{\rho}^n(x_j) + \frac{\Delta t}{2}\rho_t(t^n, x_j), \tag{53}$$

$$R(t^{n+\frac{1}{2}}, x_j) \approx R(t^n, x_j) + \frac{\Delta t}{2}R_t(t^n, x_j). \tag{54}$$

By (49) we have:

$$\tilde{\rho}^n(x_j) = \bar{\rho}^n_j$$

and by (47) we evaluate the time derivative ρ_t as:

$$\rho_t(t^n, x_j) = -F(\rho(t^n, x_j), R(t^n, x_j))_x. \tag{55}$$

The space derivative F_x in (55) is computed using the minmod limiter:

$$\begin{aligned}&F(\rho(t^n,x_j),R(t^n,x_j))_x =\\&\qquad = \text{minmod}\Bigg(\theta\frac{F(\bar\rho_j^n,R_j^n)-F(\bar\rho_{j-1}^n,R_{j-1}^n)}{\Delta x},\frac{F(\bar\rho_{j+1}^n,R_{j+1}^n)-F(\bar\rho_{j-1}^n,R_{j-1}^n)}{2\Delta x},\\&\qquad\qquad\theta\frac{F(\bar\rho_{j+1}^n,R_{j+1}^n)-F(\bar\rho_j^n,R_j^n)}{\Delta x}\Bigg).\end{aligned}$$

We compute the terms in (54) by the composite trapezoidal rule, noting that the first and the last interval are halved. Recalling that $N := \eta/\Delta x$ we get:

$$\begin{aligned}R(t^n,x_j) &= \int_{x_j}^{x_j+\eta}\tilde\rho(t^n,y)w(y-x_j)\,dy\\&\approx\left[\tilde\rho^n(x_j)w(0)+\tilde\rho^n(x_{j+\frac12})w\left(\frac{\Delta x}{2}\right)\right]\frac{\Delta x}{4}\\&\quad+\left[\tilde\rho^n(x_{j+N})w(\eta)+\tilde\rho^n(x_{j+N-\frac12})w\left(\eta-\frac{\Delta x}{2}\right)\right]\frac{\Delta x}{4}\\&\quad+\sum_{k=1}^{N-1}\left[\tilde\rho^n(x_{j+k+\frac12})w\left(\left(k+\frac12\right)\Delta x\right)+\tilde\rho^n(x_{j+k-\frac12})w\left(\left(k-\frac12\right)\Delta x\right)\right]\frac{\Delta x}{2},\end{aligned}$$

with $\tilde\rho^n(x_{j+k+\frac12})=\bar\rho^n_{j+k}+s^n_{j+k}\frac{\Delta x}{2}$ and $\tilde\rho^n(x_{j+k-\frac12})=\bar\rho^n_{j+k}-s_{j+k}\frac{\Delta x}{2}$, and

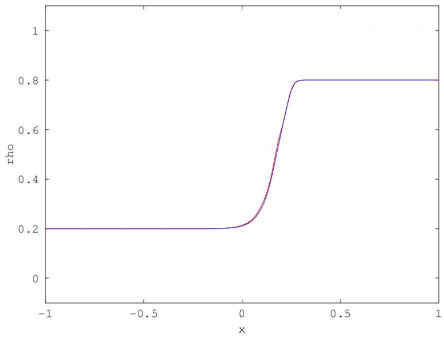
$$\begin{aligned}R_t(t^n,x_j)&=\int_{x_j}^{x_j+\eta}\rho_t(t^n,y)w(y-x_j)\,dy\\&=-\int_{x_j}^{x_j+\eta}F(\rho(t^n,y),R(t^n,y))_y\;w(y-x_j)\,dy\\&=\int_{x_j}^{x_j+\eta}F(\rho(t^n,y),R(t^n,y))\,w'(y-x_j)\,dy-\Big[F(\rho(t^n,y),R(t^n,y))\,w(y-x_j)\Big]_{y=x_j}^{y=x_j+\eta}\\&=F(\rho(t^n,x_j),R(t^n,x_j))w(0)-F(\rho(t^n,x_j+\eta),R(t^n,x_j+\eta))w(\eta)\\&\quad+\int_{x_j}^{x_j+\eta}F(\rho(t^n,y),R(t^n,y))w'(y-x_j)\,dy\\&=F(\rho(t^n,x_j),R(t^n,x_j))w(0)-F(\rho(t^n,x_j+\eta),R(t^n,x_j+\eta))w(\eta)\\&\quad+\frac{\Delta x}{2}\left[F(\rho(t^n,x_j),R(t^n,x_j))w'(0)-F(\rho(t^n,x_{j+N}),R(t^n,x_{j+N}))w'(\eta)\right]\\&\quad+\Delta x\sum_{k=1}^{N-1}F(\rho(t^n,x_{j+k}),R(t^n,x_{j+k}))w'(k\Delta x).\end{aligned}$$

Since the derived scheme uses alternating, staggered grids, we have to distinguish between the odd and even time steps. (52), (49)-(50), (53)-(54) describe the odd steps. The even steps are obtained by shifting the indexes in the aforementioned equations by $\frac12$ and the computational domain should be extended by $\frac{\Delta x}{2}$ on both sides. Concerning boundary conditions, in our numerical tests the solution is constant at the edges of the computational domain, so we use this constant values for our extended domain.

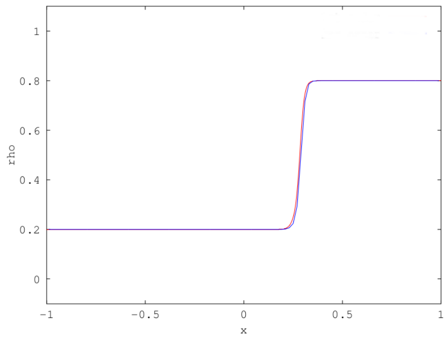
3.2 Numerical tests

We compare the solution obtained with the first order method introduced in Section 2 and the second order method (Figure 7). We consider a Riemann problem with data $\rho_L = 0.2$ and

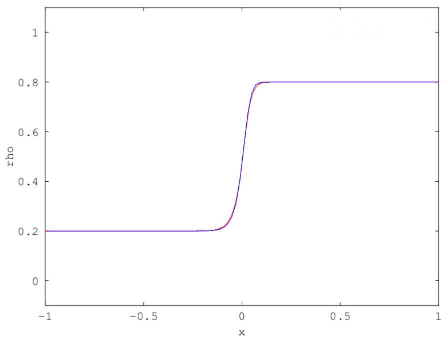
$\rho_R = 0.8$ and final time $T = 0.5$. We take $\eta = 0.1$ and $\theta = 2$. We use the space step $\Delta x = 0.002$ for the first order scheme and $\Delta x = 0.02$ for the second order scheme.



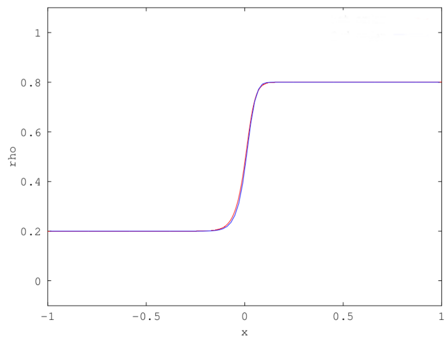
(a) Underwood's velocity with constant kernel



(b) Greenshield's velocity ($n = 5$) with constant kernel



(c) linear velocity with linear kernel



(d) linear velocity with concave kernel

Figure 7: Density profiles that solve the equation (1) with the first-order (red) and the second-order scheme (blue) at time $t = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$, $\rho_R = 0.8$.

3.2.1 Convergence orders

We compute numerical convergence orders for the second order scheme, as in the previous Section 2.7.3, for $\Delta x = 0.01, 0.005, 0.0025, 0.00125, 0.000625$. We calculate the convergence order and the $\mathbf{L^1}$-error at final time $T = 0.5$ using (44)-(45). Observe that we make a distinction between $\theta = 1$ and $\theta = 2$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$		$w_\eta(x) = 2x/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.826504	1.564052 e-03	1.035035	1.558680 e-03	-0.167287	1.000649 e-02
0.005	0.874444	8.819596 e-04	1.010809	7.606422 e-04	0.407942	1.123674 e-02
0.0025	0.926451	4.810771 e-04	0.999683	3.774822 e-04	1.070325	8.469106 e-03
0.00125	0.905707	2.531192 e-04	0.996911	1.887826 e-04	0.985812	4.033086 e-03

Table 4: Convergence orders and $\mathbf{L^1}$-errors for constant, linear decreasing and linear increasing kernel with linear decreasing velocity $v(\rho) = 1 - \rho$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$ with $\theta = 1$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$		$w_\eta(x) = 2x/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.898562	1.584519 e-03	0.999447	1.500399 e-03	0.218408	1.392697 e-02
0.005	1.028046	8.499700 e-04	0.999307	7.504870 e-04	1.315667	1.197041 e-02
0.0025	0.941087	4.168028 e-04	0.997995	3.754238 e-04	1.034576	4.808990 e-03
0.00125	0.953895	2.170876 e-04	0.997035	1.879728 e-04	1.030555	2.347552 e-03

Table 5: Convergence orders and $\mathbf{L^1}$-errors for constant, linear decreasing and linear increasing kernel with linear decreasing velocity $v(\rho) = 1 - \rho$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$ with $\theta = 2$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.496053	2.404947 e-03	0.937675	1.706889 e-03
0.005	0.664771	1.705213 e-03	0.942463	8.911211 e-04
0.0025	0.775108	1.075628 e-03	0.862285	4.636893 e-04
0.00125	0.796789	6.285382 e-04	0.865828	2.550663 e-04

Table 6: Convergence orders and $\mathbf{L^1}$-errors for constant and linear decreasing kernel with Underwood's velocity $v(\rho) = e^{-\rho}$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$ with $\theta = 1$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.539712	2.309898 e-03	0.904230	1.558415 e-03
0.005	1.031363	1.588997 e-03	0.901378	8.326884 e-04
0.0025	0.837733	7.774127 e-04	0.884343	4.458005 e-04
0.00125	0.900772	4.349795 e-04	0.974746	2.415053 e-04

Table 7: Convergence orders and $\mathbf{L^1}$-errors for constant and linear decreasing kernel with Underwood's velocity $v(\rho) = e^{-\rho}$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$ with $\theta = 2$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.726107	1.498004 e-03	0.638340	1.533601 e-03
0.005	1.243070	9.055925 e-04	1.398245	9.852642 e-04
0.0025	0.908877	3.825881 e-04	0.831466	3.737996 e-04
0.00125	1.006300	2.037661 e-04	1.035231	2.100596 e-04

Table 8: Convergence orders and $\mathbf{L^1}$-errors for constant and linear decreasing kernel with Greenshield's velocity $v(\rho) = 1 - \rho^5$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$ with $\theta = 1$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.870731	1.502205 e-03	0.838007	1.546337 e-03
0.005	1.090038	8.215106 e-04	1.202457	8.650452 e-04
0.0025	0.882361	3.859036 e-04	0.877248	3.758918 e-04
0.00125	0.979493	2.093445 e-04	0.887723	2.046373 e-04

Table 9: Convergence orders and $\mathbf{L^1}$-errors for constant and linear decreasing kernel with Greenshield's velocity $v(\rho) = 1 - \rho^5$ at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.2$ and $\rho_R = 0.8$ with $\theta = 2$.

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ISSN 0249-6399